ON THE RING OF COOPERATIONS FOR REAL HERMITIAN K-THEORY

JACKSON MORRIS

ABSTRACT. Let kq denote the very effective cover of the motivic Hermitian K-theory spectrum. We analyze the ring of cooperations $\pi_{**}^{\mathbb{R}}(\mathrm{kq} \otimes \mathrm{kq})$ in the stable motivic homotopy category $\mathrm{SH}(\mathbb{R})$, giving a full description in terms of Brown–Gitler comodules. To do this, we decompose the E_2-page of the motivic Adams spectral sequence and show that it must collapse. The description of the E_2-page is accomplished by a series of algebraic Atiyah–Hirzebruch spectral sequences which converge to the summands of the E_2-page. Along the way, we prove a splitting result for the very effective symplectic K-theory ksp over any base field of characteristic not two.

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1. INTRODUCTION

1.1. **Motivation.** In stable homotopy theory, one of the most powerful tools we have for computing the homotopy groups of spheres is the Adams spectral sequence. For a ring spectrum E, the E-Adams spectral sequence (E-**ASS**) has signature

$$E_1^{s,f} = \pi_{s+f}(\mathrm{E} \otimes \overline{\mathrm{E}}^{\otimes f}) \implies \pi_s(\mathbb{S}_{\mathrm{E}}^{\wedge}),$$

where $\overline{\mathrm{E}}$ is the cofiber of the unit map $\mathbb{S} \to \mathrm{E}$. Under mild assumptions on E, this spectral sequence strongly converges to the homotopy groups of $\mathbb{S}_{\mathrm{E}}^{\wedge}$, the E-nilpotent completion of the sphere spectrum [Rav86]. To begin to compute with the E-**ASS**, one must understand the smash powers $\mathrm{E}^{\otimes f}$.

Let bo denote the connective cover of the real topological K-theory spectrum KO. This spectrum is well-enough behaved that one may begin to study its associated Adams spectral sequence. Mahowald and others have studied the bo-**ASS**, known as the bo-resolution, at the prime 2 extensively [Mah81; LM87; Bea+20]; their analysis of this spectral sequence leads to, among many things, the v_1-torsion

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free component of $\pi_* \mathbb{S}$. Key to Mahowald's analysis is a spectrum level splitting of $\mathrm{bo} \otimes \mathrm{bo}$ as the following wedge sum:

$$\mathrm{bo} \otimes \mathrm{bo} \simeq \bigoplus_{k \geq 0} \Sigma^{4k} \mathrm{bo} \otimes \mathrm{H}\mathbb{Z}_k^{cl}.$$

The spectra $\mathrm{H}\mathbb{Z}_k^{cl}$ are finite complexes known as the integral Brown–Gitler spectra [BG73; GJM86].

In stable motivic homotopy theory, a primary object of investigation is the homotopy ring of the motivic sphere spectrum. This ring has rich connections with the Grothendieck-Witt ring of symmetric bilinear forms [Mor12], Milnor K-theory [BBX25], and Hermitian K-theory [RSØ19; RSØ24]. Similar to the classical case, one of the most powerful and well-studied tools is the motivic Adams spectral sequence. For a motivic ring spectrum E, there is an E-motivic Adams spectral sequence (E-**mASS**) which can be used to compute these homotopy groups that, under mild conditions, converges strongly to the homotopy groups of the E-nilpotent completion $\mathbb{S}_{\mathrm{E}}^{\wedge}$ [DI10; HKO11]. This takes the form

$$E_1^{s,f,w} = \pi_{s+f,w}^F(\mathrm{E} \otimes \overline{\mathrm{E}}^{\otimes f}) \implies \pi_{s,w}^F(\mathbb{S}_{\mathrm{E}}^{\wedge}).$$

To begin to compute with the E-**mASS**, one must understand the smash powers $\mathrm{E}^{\otimes f}$.

Let kq denote the very effective cover of the Hermitian K-theory spectrum KQ [ARØ20] (see Section 3 for details on the very effective slice filtration). In [CQ21], Culver–Quigley analyze the kq-**mASS** at the prime 2 in $\mathbb{C}$-motivic homotopy theory. This spectral sequence is called the kq-resolution and has signature

$$E_1^{s,f,w} = \pi_{s+f,w}^{\mathbb{C}}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes f}) \implies \pi_{s,w}^{\mathbb{C}}(\mathbb{S}_{\mathrm{kq}}^{\wedge}).$$

Their analysis of this spectral sequence leads to, among many things, the v_1-torsion free component of $\pi_{**}^{\mathbb{C}}\mathbb{S}$. Toward computing the E_1-page, Culver–Quigley compute the ring of cooperations $\pi_{**}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$ by the $\mathrm{H}\mathbb{F}_2$-**mASS**$(\mathrm{kq} \otimes \mathrm{kq})$ which takes the form

$$E_2^{s,f,w} = \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}^{s,f,w}\left(\mathbb{M}_2^{\mathbb{C}}, \mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq})\right) \implies \pi_{**}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq}).$$

Here, $\mathcal{A}_{\mathbb{C}}^{\vee}$ denotes the $\mathbb{C}$-motivic dual Steenrod algebra, $\mathrm{H}_{**}(-)$ denotes mod-2 motivic homology, and $\mathbb{M}_2^{\mathbb{C}}$ denotes the mod-2 motivic homology of a point. We implicitly 2-complete the target of this spectral sequence, as we will throughout this paper, to ensure convergence.

1.2. **Main results.** The goal of this paper is to begin the study of the kq-resolution in $\mathbb{R}$-motivic homotopy theory. As a step in this direction, we compute the ring of cooperations $\pi_{**}^{\mathbb{R}}(\mathrm{kq} \otimes \mathrm{kq})$ as a module over $\pi_{**}^{\mathbb{R}}(\mathrm{kq})$.

Theorem A (Theorem 4.1, Theorem 8.8). *There is an isomorphism*

$$\pi_{**}^{\mathbb{R}}(\mathrm{kq} \otimes \mathrm{kq}) \cong \bigoplus_{k \geq 0} \Sigma^{4k,2k} \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(\mathbb{M}_2^{\mathbb{R}}, B_0^{\mathbb{R}}(k)\right),$$

where $B_0^{\mathbb{R}}(k)$ denotes the k-th integral motivic Brown–Gitler comodule over $\mathbb{R}$ (see Section 2.3). We describe the right-hand completely, modulo v_1-torsion.

This result gives substantial control over the n-line of the E_1-page of the kq resolution; see Remark 8.9. The reader familiar with Mahowald's analysis of the bo-resolution [Mah81] may be inclined to believe that this result may be proven in a similar fashion. This is not the case. Mahowald's arguments make great use of integral Brown–Gitler spectra. However, there is currently no known construction of motivic integral Brown–Gitler spectra except in very particular cases (see Example 2.16, Remark 2.17), and so another method of analysis must be employed. As is the case over $\mathbb{C}$, we generally follow the strategy employed by [Beh+19], in which they compute the classical ring of cooperations for tmf using algebraic methods. We outline this strategy below.

(1) We first introduce the integral motivic Brown–Gitler comodules $B_0^{\mathbb{R}}(k)$ and show that there are isomorphisms of $\mathcal{A}(1)_{\mathbb{R}}^{\vee}$-comodules:

$$\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} \mathbin{/\mkern-6mu/} \mathcal{A}(1))_{\mathbb{R}}^{\vee} \cong \bigoplus_{k \geq 0} \Sigma^{4k,2k} B_0^{\mathbb{R}}(k).$$

(2) Next, we produce short exact sequences relating the integral Brown–Gitler comodules which allow for induction.

(3) Last, we make base-case computations in Ext using algebraic Atiyah–Hirzebruch spectral sequences (**aAHSS**) and induct to complete the proof of the theorem.

Steps (1) and (2) are direct consequences of [CQ21, Section 3]. Step (3) is the most difficult and occupies most of this paper. This is because the algebra $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ has more elaborate structure than its classical and $\mathbb{C}$-motivic counterparts. Indeed, when presenting the charts for our spectral sequence computations, we find it best to follow the strategy of Guillou–Hill–Isaksen–Ravenel [Gui+20] and organize each page by coweight $cw = stem - weight$. There is an element $\tau^4 \in \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ which gives a periodicity allowing us to organize our data into 4 sets of charts. These computation are manageable with careful bookkeeping.

Through this process, we obtain the following result, where ksp denotes the very effective cover of $\Sigma^{4,2}\mathrm{KQ}$ (see Section 3):

Theorem B (Corollary 7.4). *There is an isomorphism:*

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}, B_0^{\mathbb{R}}(1)) \cong \pi_{**}^{\mathbb{R}}(\mathrm{ksp}).$$

This is an easy consequence of a stronger result which we show in a more general setting.

Theorem C (Proposition 3.7). *Let F be any field with 2 invertible. Then there is an equivalence of motivic spectra*

$$\mathrm{ksp} \simeq \mathrm{H}\mathbb{Z}_1^F \otimes \mathrm{kq},$$

where $\mathrm{H}\mathbb{Z}_1^F$ denotes the first integral F-motivic Brown–Gitler spectrum (see Example 2.16).

We prove this result by analyzing the very effective slice tower for KQ similar to how one would analyze the Whitehead tower for the classical spectrum KO.

1.3. **Future directions.** We next outline several directions in which we will extend the results of this work.

The kq-*resolution.* This paper initiates the study of the kq-resolution in $\mathrm{SH}(\mathbb{R})$. In particular, we calculate the E_1-page. While many of the differentials in this spectral sequence can be determined using the base change functor

$$- \otimes \mathbb{C} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(\mathbb{C}),$$

there is ρ-torsion which must be determined using the C_2-equivariant Betti realization functor

$$\mathrm{Be}^{C_2} : \mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}^{C_2}$$

along with information involving the ko_{C_2}-resolution. We plan to study this in future work. Additionally, the kq-resolution is intricately related to v_1-periodic homotopy. The work of Belmont–Isaksen–Kong [BIK24] determines a version of v_1-periodic homotopy theory in $\pi_{**}^{\mathbb{R}}\mathbb{S}$ by computing the homotopy groups $\pi_{**}^{\mathbb{R}}\mathrm{L}$, where L sits in a cofiber sequence

$$\mathrm{L} \to \mathrm{kq} \xrightarrow{\psi^3 - 1} \mathrm{kq}.$$

It will be interesting to compare the homotopy groups of $\pi_{**}^{\mathbb{R}}\mathrm{L}$ with the v_1-periodic homotopy detected by the kq-resolution (see also Remark 3.8).

Connections with C_2-equivariant homotopy theory. In C_2-equivariant homotopy theory, the work of Li–Petersen–Tatum [LPT24] produces C_2-equivariant analogues of the classical integral Brown–Gitler spectra. These are spectra $H\mathbb{Z}_k^{C_2}$ such that, as comodules over the the sub-Hopf algebra $\mathcal{A}(1)_{C_2}^\vee$ of the C_2-equivariant dual Steenrod algebra, there is an isomorphism

$$H_\star(H\mathbb{Z}_k^{C_2}) \cong B_0^{C_2}(k),$$

where $B_0^{C_2}(k)$ is the k-th C_2-equivariant integral Brown–Gitler comodule. It was recently shown [LPT25] that there is a splitting of $\mathrm{ku}_\mathbb{R}$-module spectra:

$$\mathrm{ku}_\mathbb{R} \otimes \mathrm{ku}_\mathbb{R} \simeq \bigoplus_{k \geq 0} \Sigma^{\rho k}\mathrm{ku}_\mathbb{R} \otimes H\mathbb{Z}_k^{C_2}.$$

One can ask for a similar splitting of the C_2-spectrum ko_{C_2}. In particular, it is conjectured that there is a splitting of ko_{C_2}-module spectra

$$\mathrm{ko}_{C_2} \otimes \mathrm{ko}_{C_2} \simeq \bigoplus_{k \geq 0} \Sigma^{2\rho k}\mathrm{ko}_{C_2} \otimes H\mathbb{Z}_k^{C_2}.$$

One way to show that such a splitting exists is to construct maps at the level of homology

$$\theta_k : \Sigma^{2\rho k}H_\star(H\mathbb{Z}_k^{C_2}) \to H_\star(\mathrm{ko}_{C_2})$$

and show that they survive the ko_{C_2}-relative Adams spectral sequence

$$E_2 = \mathrm{Ext}_{\mathcal{A}(1)_{C_2}^\vee}\left(H_\star(\Sigma^{2\rho k}H\mathbb{Z}_k^{C_2}), H_\star(\mathrm{ko}_{C_2})\right) \implies [\Sigma^{2\rho k}\mathrm{ko}_{C_2} \otimes H\mathbb{Z}_k^{C_2}, \mathrm{ko}_{C_2} \otimes \mathrm{ko}_{C_2}]^{\mathrm{ko}_{C_2}}.$$

It can be shown that the E_2-page of this spectral sequence decomposes into terms of the form

$$\mathrm{Ext}_{\mathcal{A}(1)_{C_2}^\vee}(B_0^{C_2}(k), B_0^{C_2}(m)).$$

This further splits as

$$\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}(B_0^\mathbb{R}(k), B_0^\mathbb{R}(m)) \oplus \mathrm{Ext}_{NC}(B_0^{C_2}(k), B_0^{C_2}(m)),$$

where Ext_{NC} refers to Ext over the negative cone. Our computations in this paper offer progress towards the $\mathbb{R}$-motivic Ext groups in this decomposition. These ideas will be further investigated in future work, joint with Li, Petersen, and Tatum.

1.3.1. *The ring of cooperations over general bases.* The work of Kylling [Kyl15] determines the group $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{F}_3}^\vee}(\mathbb{M}_2^{\mathbb{F}_3}, \mathbb{M}_2^{\mathbb{F}_3})$. In forthcoming work, we use the methods developed in this paper to determine the ring of cooperations $\pi_{\star\star}^{\mathbb{F}_3}(\mathrm{kq} \otimes \mathrm{kq})$. A remarkable result of Bachmann–Østvær [BØ22] gives a useful pullback square that allows us to make a deeper statement. Namely, if $E \in \mathrm{SH}(\mathbb{Z}[1/2])^{\mathrm{cell}}$, then there is a pullback square

$$\begin{array}{ccc}
\mathrm{map}_{\mathbb{Z}[1/2]}(\mathbb{S}, E)_2^\wedge & \longrightarrow & \mathrm{map}_\mathbb{R}(\mathbb{S}, E)_2^\wedge \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{map}_{\mathbb{F}_3}(\mathbb{S}, E)_2^\wedge & \longrightarrow & \mathrm{map}_\mathbb{C}(\mathbb{S}, E)_2^\wedge
\end{array}$$

where $\mathrm{map}_F(\mathbb{S}, E)$ denotes the ordinary spectrum of maps between $\mathbb{S}$ and E in $\mathrm{SH}(F)$ and and the morphisms are a combination of base change and Brauer liftings. The computation of $\pi_{\star\star}^F(\mathrm{kq} \otimes \mathrm{kq})$ for $F = \mathbb{C}, \mathbb{R}, \mathbb{F}_3$ allows one to access the arithmetic invariant $\pi_{\star\star}^{\mathbb{Z}[1/2]}(\mathrm{kq} \otimes \mathrm{kq})$. This bypasses the fact that, although we understand the dual motivic Steenrod algebra $\mathcal{A}_{\mathbb{Z}[1/2]}^\vee$ (see [Spi18, Theorem 11.24], also [DØ23, Section 4.3]), there is currently no description of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{Z}[1/2]}^\vee}(\mathbb{M}_2^{\mathbb{Z}[1/2]})$.

1.4. **Organization.** Part 1 of this paper presents background and recalls the work of Culver–Quigley in [CQ21]. In Section 2, we recall relevant facts about the stable motivic homotopy category $SH(F)$, the kq-resolution, and motivic Brown–Gitler comodules. In Section 3, we prove a novel result about symplectic K-theory. In Section 4, we outline a process for computing the ring of cooperations $\pi_{**}^F(kq \otimes kq)$. Much of the material of these section is contained [CQ21, Section 3], but we phrase things in slightly more generality, which will be useful to a broader audience. In Section 5, we review the computation of $\pi_{**}^{\mathbb{C}}(kq \otimes kq)$ as a warm up for the main contents of our work.

Part 2 of this paper contains the computation of the ring of cooperations $\pi_{**}^{\mathbb{R}}(kq \otimes kq)$ and applications. In Section 6, we give describe the ring $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ and some modules which will frequently appear. In Section 7, we compute the ring $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$ by a series of algebraic Atiyah–Hirzebruch spectral sequences. In Section 8, we compute the ring of cooperations $\pi_{**}^{\mathbb{R}}(kq \otimes kq)$ and apply our results to describe the E_1-page of the kq-resolution.

In Appendix A, we present a set of charts representing the modules described in Section 6.

1.5. **Notation and Conventions.** We use the following notation throughout.

- $H\mathbb{F}_2$ is the motivic Eilenberg-Mac Lane spectrum representing mod-2 motivic cohomology.
- $H_{**}(X)$ is the mod-2 motivic homology of X.
- $\pi_{**}^F(X)$ are the bigraded motivic homotopy groups of X. These are suitably completed so that the motivic Adams spectral sequence always converges.
- $\mathbb{M}_2^F$ is the mod-2 motivic homology of a point.
- $\mathcal{A}_F^{\vee}$ denotes the F-motivic dual motivic Steenrod algebra.
- $\mathcal{A}(n)_F^{\vee}$ denotes the dual of the subalgebra of the F-motivic Steenrod algebra generated by $Sq^1, Sq^2, \ldots, Sq^{2^n}$.
- KQ denotes the motivic spectrum representing Hermitian K-theory, kq denotes its very effective cover, and ksp denotes the very effective cover of $\Sigma^{4,2}KQ$
- $\mathrm{Ext}_B(\mathbb{M}_2^F, X)$ will be abbreviated by $\mathrm{Ext}_B(X)$.
- We grade Ext groups as (s, f, w), where s is the stem, f is the Adams filtration, and w is the motivic weight. We also let $cw = s - w$ denote the coweight, sometimes referred to in the literature as the Milnor-Witt degree.
- All charts are in Adams grading, meaning (s, f), with motivic weight suppressed.
- All motivic spectra are implicitly 2-complete unless explictly stated otherwise.
- For E a motivic ring spectrum, we let E-**mASS**(X) denote the motivic Adams spectral sequence based on E and converging to $\pi_{**}^F(X_E^{\wedge})$. When E $= H\mathbb{F}_2$, we denote this spectral sequence by **mASS**(X).
- For M an $\mathcal{A}(1)_F^{\vee}$-comodule, we let **aAHSS**(M) denote the algebraic Atiyah–Hirzebruch spectral sequence detailed in Section 4.2.
- We employ two algebraic shift functors, denoted $\Sigma^{p,q}(-)$ and $(-)\langle n \rangle$. For M a module over Ext and $x \in M$ an element of degree (s, f, w), we have

$$|\Sigma^{p,q}x| = (s + p, f, w + q)$$

and

$$|x\langle n \rangle| = (s, f + n, w).$$

Part 1. **Preliminaries**

2. Stable motivic homotopy theory

In this section we work over an arbitrary field F of characteristic different from 2 such that $\mathrm{vcd}_2(F) < \infty$. We recall the kq-resolution then review the dual motivic Steenrod algebra and motivic Brown–Gitler comodules.

2.1. **The kq-resolution.** It was shown in [Hor05] that there is an $(8,4)$-periodic motivic ring spectrum $\mathrm{KQ} \in \mathrm{SH}(F)$ representing Hermitian K-theory, in that $\pi_{s,w}^F \mathrm{KQ} = \mathrm{K}_{s-2w}^{\mathrm{h}}(F)$. In particular, $\pi_{8n,4n}^F \mathrm{KQ} = \mathrm{GW}(F)$, the Grothendieck–Witt ring of quadratic forms over F, and the unit map $\mathbb{S} \to \mathrm{KQ}$ induces Morel's isomorphism [RSØ19]

$$\pi_{-n,-n}^F \mathbb{S} \cong \mathrm{K}_n^{\mathrm{MW}}(F).$$

Let kq denote the very effective cover of KQ [ARØ20] (see Section 3 for more details on the very effective filtration). There is a canonical Adams tower associated to the unit map $\mathbb{S} \to \mathrm{kq}$:

$$
\begin{array}{ccccccc}
\mathbb{S} & \longleftarrow & \Sigma^{-1,0}\overline{\mathrm{kq}} & \longleftarrow & \Sigma^{-2,0}\overline{\mathrm{kq}} \otimes \overline{\mathrm{kq}} & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{kq} & & \Sigma^{-1,0}\overline{\mathrm{kq}} \otimes \mathrm{kq} & & \Sigma^{-2,0}\overline{\mathrm{kq}} \otimes \overline{\mathrm{kq}} \otimes \mathrm{kq} & &
\end{array}
$$

Applying $\pi_{**}^F(-)$ yields the motivic Adams spectral sequence kq-**mASS**($\mathbb{S}$) known as the kq-resolution. Its properties were first studied by Culver–Quigley [CQ21]. We remark that we are working in the 2-complete category.

Theorem 2.1. *[CQ21, Thm 2.1] The* kq-*resolution is a strongly convergent spectral sequence of the form*

$$E_1^{s,f,w} = \pi_{s+f,w}^F(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes f}) \implies \pi_{s,w}^F \mathbb{S}_2^\wedge.$$

The d_r-differentials have the form

$$d_r : E_r^{s,f,w} \to E_r^{s-1,f+r,w}.$$

Remark 2.2. We can also look at the kq-resolution in the integral, uncompleted setting. The unit map $\mathbb{S} \to \mathrm{kq}$ gives an isomorphism $\Pi_0^F(\mathrm{kq}) \cong \Pi_0^F(\mathbb{S})$, where $\Pi_n^F(X) := \bigoplus_{k \in \mathbb{Z}} \pi_{n+k,k}^F X$ denotes the k-th coweight stem (also known as the k-th Milnor–Witt stem). This implies that $\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}$ is n-connected in Morel's homotopy t-structure. In fact, this is enough to show that $\mathbb{S}_{\mathrm{kq}}^\wedge \simeq \mathbb{S}$.

However, we will compute this E_1-page by a motivic Adams spectral sequence based on the Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbb{F}_2$. This spectral sequence computes the homotopy groups of the $(2,\eta)$-completion of $\mathrm{kq} \otimes \mathrm{kq}$. When F is a field of finite virtual 2-cohomological dimension, then this coincides with the 2-completion [HKO11, Theorem 1]. These conventions force us to work in the 2-complete category (see also [CQ21, Section 2.2]).

Let bo denote the connective cover of the classical real K-theory spectrum KO. The kq-resolution is the motivic analogue of the bo-resolution in the following way. The E_1-page of the bo-resolution is of the form

$$E_1^{s,f} = \pi_{s+f}(\mathrm{bo} \otimes \overline{\mathrm{bo}}^{\otimes f}) \implies \pi_* \mathbb{S}_2^\wedge,$$

where we may identify $\mathbb{S}_{\mathrm{bo}}^\wedge \simeq \mathbb{S}_2^\wedge$ in the 2-complete setting [Bou79]. There is a *complex Betti realization functor*

$$\mathrm{Be}^{\mathbb{C}} : \mathrm{SH}(\mathbb{C}) \to \mathrm{Sp},$$

induced by the analytic topology on a scheme $X \in \mathrm{Sm}_{\mathbb{C}}$ after taking $\mathbb{C}$-points. Moreover, if F admits an embedding into $\mathbb{C}$, then there is complex realization $\mathrm{Be}^{\mathbb{C}} : \mathrm{SH}(F) \to \mathrm{Sp}$ defined in the same way. The following result lets us compare between kq and bo:

Theorem 2.3. *[ARØ20, Lemma 2.13] If F admits a complex embedding, then there is an equivalence of spectra*

$$\mathrm{Be}^{\mathbb{C}}(\mathrm{kq}) \simeq \mathrm{bo}.$$

Betti realization is symmetric monoidal and exact [HO16], giving an immediate corollary:

Corollary 2.4. *[CQ21, Cor 2.7] Betti realization sends the* kq*-resolution to the* bo*-resolution. In particular, there is a multiplicative map of spectral sequences:*

$$E_r^{s,f,*} \to E_r^{s,f}.$$

Remark 2.5. Note that, just as in the classical case, the spectrum kq is not flat in the sense of Adams. Thus the E_1-page does not identify with the cobar complex computing the Hopf algebroid cohomology $\mathrm{Ext}_{\pi_{**}^F \mathrm{kq} \otimes \mathrm{kq}}(\pi_{**}^F \mathrm{kq}, \pi_{**}^F \mathrm{kq})$, and so we must analyze the E_1-page of the kq-resolution.

Remark 2.6. There is a complex Betti realization functor in the case of $F = \mathbb{R}$ to which we may apply Theorem 2.3. It is important to note that there is an obvious *real Betti realization*

$$\mathrm{Be}^{\mathbb{R}} : \mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}$$

which is induced by the real analytic topology on a scheme $X \in \mathrm{Sm}_{\mathbb{R}}$ after taking $\mathbb{R}$-points. These functors behave very differently. For example, consider kq $\in \mathrm{SH}(\mathbb{R})$. It was shown in [Ban25, Proposition 4.1] that $\pi_*(\mathrm{Be}^{\mathbb{R}}(\mathrm{kq})) \cong \mathbb{Z}[x]$, where $x \in \pi_4(\mathrm{Be}^{\mathbb{R}}(\mathrm{kq}))$. In particular, $\mathrm{Be}^{\mathbb{C}}(\mathrm{kq}) \not\simeq \mathrm{Be}^{\mathbb{R}}(\mathrm{kq})$.

2.2. **The dual motivic Steenrod algebra.** Let $\mathbb{M}_2^F$ denote the mod-2 motivic homology of a point. There is an isomorphism due to Voevodsky [Voe03a]:

$$\mathbb{M}_2^F = (\mathrm{K}_*^{\mathrm{M}}(F)/2)[\tau],$$

where $\mathrm{K}_*^{\mathrm{M}}(F)$ denotes the Milnor K-theory of F and $|\tau| = (0, -1)$. Recall the dual motivic Steenrod algebra:

Theorem 2.7. *[Voe03b, Section 12] The dual motivic Steenrod algebra $\mathcal{A}_F^\vee = \pi_{**}^F(\mathrm{H}\mathbb{F}_2 \otimes \mathrm{H}\mathbb{F}_2)$ is given by*

$$\mathcal{A}_F^\vee = \mathbb{M}_2^F[\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\tau}_0, \bar{\tau}_1, \ldots]/(\bar{\tau}_i^2 = \rho\bar{\tau}_{i+1} + \rho\bar{\tau}_0\bar{\xi}_{i+1} + \tau\bar{\xi}_{i+1}),$$

where $|\bar{\xi}_i| = (2^{i+1} - 2, 2^i - 1)$ and $|\bar{\tau}_i| = (2^{i+1} - 1, 2^i - 1)$. There are structure maps

$$\eta_L, \eta_R : \mathbb{M}_2^F \to \mathcal{A}_F^\vee, \quad \Delta : \mathcal{A}_F^\vee \to \mathcal{A}_F^\vee \otimes_{\mathbb{M}_2^F} \mathcal{A}_F^\vee$$

which are determined by the following formulae:

$$
\begin{aligned}
\eta_L(\rho) &= \rho & \eta_L(\tau) &= \tau; \\
\eta_R(\rho) &= \rho & \eta_R(\tau) &= \tau + \bar{\tau}_0\rho; \\
\Delta(\bar{\xi}_k) &= \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i}, & \Delta(\bar{\tau}_i) &= \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{2^i} + 1 \otimes \bar{\tau}_k.
\end{aligned}
$$

We will also encounter various subalgebras and quotient algebras of $\mathcal{A}_F^\vee$.

Definition 2.8. For $n \geq 0$, let $\mathcal{A}(n)_F^\vee$ be the quotient algebra

$$\mathcal{A}(n)_F^\vee \cong \mathcal{A}_F^\vee/(\bar{\xi}_1^{2^n}, \bar{\xi}_2^{2^{n-1}}, \ldots, \bar{\xi}_n^2, \bar{\xi}_{n+1}, \bar{\xi}_{n+2}, \ldots, \bar{\tau}_{n+1}, \bar{\tau}_{n+2}, \ldots).$$

There is a sub-Hopf algebra of the motivic Steenrod algebra

$$\mathcal{A}(n)_F = \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^n} \rangle;$$

let $(\mathcal{A} /\!/ \mathcal{A}(n))_F^\vee$ be the subalgebra of $\mathcal{A}_F^\vee$ given by

$$(\mathcal{A} /\!/ \mathcal{A}(n))_F^\vee = \mathbb{M}_2^F[\bar{\xi}_1^{2^n}, \bar{\xi}_2^{2^{n-1}}, \ldots, \bar{\tau}_{n+1}, \ldots]/(\bar{\tau}_i^2 = \rho\bar{\tau}_{i+1} + \rho\bar{\tau}_0\bar{\xi}_{i+1} + \tau\bar{\xi}_{i+1}).$$

Remark 2.9. An important distinction between $\mathcal{A}_{\mathbb{R}}^{\vee}$ and $\mathcal{A}_{\mathbb{C}}^{\vee}$ is that while the latter is a Hopf algebra, the former is a Hopf algebroid [Rav86, Appendix A]. There is a canonical class $\rho \in \mathrm{K}_1^{\mathrm{M}}(F)/2$ representing -1, giving rise to a class in $\mathbb{M}_2^F$. If -1 is a square in F, then this class is trivial in motivic homology and $\mathcal{A}_F^{\vee}$ is an honest Hopf algebra. If -1 is not a square, then there is an action of the motivic Steenrod algebra:

$$\mathrm{Sq}^1(\tau) = \rho.$$

In particular, upon dualizing we see that $\mathbb{M}_2^F$ is not central in $\mathcal{A}_F^{\vee}$, implying that $(\mathbb{M}_2^F, \mathcal{A}_F^{\vee})$ is a Hopf algebroid.

As in the classical case, one can express the sub-Hopf algebroid $(\mathcal{A} /\!/ \mathcal{A}(n))_F^{\vee}$ as a cotensor product:

$$(\mathcal{A} /\!/ \mathcal{A}(n))_F^{\vee} = \mathcal{A}_F^{\vee} \square_{\mathcal{A}(n)_F^{\vee}} \mathbb{M}_2^F.$$

These sub-Hopf algebroids come up naturally in our work as the motivic homology of certain motivic spectra. For instance, we have that $\mathrm{H}_{**}(\mathrm{H}\mathbb{Z}) \cong (\mathcal{A} /\!/ \mathcal{A}(0))_F^{\vee}$. We also have the following:

Theorem 2.10. *[ARØ20, Remark 2.14] There is an isomorphism of $\mathcal{A}_F^{\vee}$-comodule algebras*

$$\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} /\!/ \mathcal{A}(1))_F^{\vee} = \mathbb{M}_2^F[\bar{\xi}_1^2, \bar{\xi}_2, \dots, \bar{\tau}_2, \bar{\tau}_3, \dots]/(\bar{\tau}_i^2 = \rho\bar{\tau}_{i+1} + \rho\bar{\tau}_0\bar{\xi}_{i+1} + \tau\bar{\xi}_{i+1}).$$

We investigate now the $\mathcal{A}_F^{\vee}$-comodule structure on the homology $\mathrm{H}_{**}(\mathrm{kq})$. As $(\mathcal{A} /\!/ \mathcal{A}(1))_F^{\vee} \subseteq \mathcal{A}_F^{\vee}$ is a subalgebra, the coaction comes from the restriction of the comultiplication on $\mathcal{A}_F^{\vee}$:

$$\psi : \mathrm{H}_{**}(\mathrm{kq}) \to \mathcal{A}_F^{\vee} \otimes_{\mathbb{M}_2^F} \mathrm{H}_{**}(\mathrm{kq}).$$

Later, we will need to understand the $\mathcal{A}(1)_F^{\vee}$-comodule structure on the homology $\mathrm{H}_{**}(\mathrm{kq})$. Observe that by definition, $\mathcal{A}(1)_F^{\vee}$ takes the form

$$\mathcal{A}(1)_F^{\vee} = \mathbb{M}_2^F[\bar{\xi}_1, \bar{\tau}_0, \bar{\tau}_1]/(\bar{\tau}_0^2 = \rho\bar{\tau}_1 + \rho\bar{\tau}_0\bar{\xi}_1 + \tau\bar{\xi}_1, \bar{\tau}_1^2, \bar{\xi}_1^2).$$

Since $\mathcal{A}(1)_F^{\vee}$ is a quotient of $\mathcal{A}_F^{\vee}$, we can deduce that the coaction is given by projection of the usual coaction of $\mathcal{A}_F^{\vee}$ on $\mathrm{H}_{**}(\mathrm{kq})$:

$$\psi : \mathrm{H}_{**}(\mathrm{kq}) \to \mathcal{A}_F^{\vee} \otimes_{\mathbb{M}_2^F} \mathrm{H}_{**}(\mathrm{kq}) \to \mathcal{A}(1)_F^{\vee} \otimes_{\mathbb{M}_2^F} \mathrm{H}_{**}(\mathrm{kq}).$$

Moreover, since kq is a ring spectrum, its homology is a comodule algebra. Thus it suffices to determine the coaction on the generators. The following is immediate from this discussion.

Proposition 2.11. *[CQ21, Corollary 3.6] The $\mathcal{A}(1)_F^{\vee}$-coaction on $\mathrm{H}_{**}(\mathrm{kq})$ is determined by the following formulae:*

$$\begin{aligned}
\psi(\bar{\xi}_1^2) &= 1 \otimes \bar{\xi}_1^2; \\
\psi(\bar{\xi}_i) &= 1 \otimes \bar{\xi}_i + \bar{\xi}_i \otimes \bar{\xi}_{i-1}^2 & i \geq 2; \\
\psi(\bar{\tau}_i) &= 1 \otimes \bar{\tau}_i + \bar{\tau}_0 \otimes \bar{\xi}_i + \bar{\tau}_1 \otimes \bar{\xi}_{i-1}^2 & i \geq 2.
\end{aligned}$$

The following result will be useful in the sequel.

Proposition 2.12. *There is a Künneth isomorphism of $\mathcal{A}_F^{\vee}$-comodule algebras*

$$\mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq}) \cong \mathrm{H}_{**}(\mathrm{kq}) \otimes_{\mathbb{M}_2^F} \mathrm{H}_{**}(\mathrm{kq}).$$

Proof. It was shown by Voevodsky [Voe03b] that the motivic Steenrod algebra $\mathcal{A}_F$ is free as a module over the cohomology of a point. One basis for this algebra is the motivic Milnor basis [Kyl17]. Since both Sq^1 and Sq^2 are elements of this basis, the quotient $\mathcal{A} /\!/ \mathcal{A}(1)_F \cong \mathcal{A}_F \otimes_{\mathcal{A}(1)_F} \mathbb{M}_2^F$ is also free over the cohomology of a point. Consider the following Künneth spectral sequence [DI05]:

$$E_2 = \mathrm{Tor}^{\mathbb{M}_2^F}(\mathrm{H}_{**}(\mathrm{kq}), \mathrm{H}_{**}(\mathrm{kq})) \implies \mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq}).$$

Since $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} /\!/ \mathcal{A}(1))_F^{\vee}$ is the $\mathbb{M}_2^F$-linear dual of a free module, it is also free. Thus the higher Tor vanishes and the spectral sequence collapses with no extensions, giving the result.   □

2.3. **Motivic Brown–Gitler comodules.** We next introduce motivic Brown–Gitler comodules. We define the *Mahowald weight filtration* on $\mathcal{A}_F^\vee$ by setting

$$wt(\bar{\xi}_i) = wt(\bar{\tau}_i) = 2^i, \quad wt(\tau) = wt(\rho) = 0$$

and letting $wt(xy) = wt(x) + wt(y)$. This naturally extends to the subalgebras $(\mathcal{A} /\!/ \mathcal{A}(n))_F^\vee$. The *F-motivic Brown–Gitler comodule* $B_n^F(k)$ is the $A(n)_F^\vee$-comodule

$$B_n^F(k) = \langle x \in (\mathcal{A} /\!/ \mathcal{A}(n))_F^\vee : wt(x) \leq 2^{n+1}k \rangle.$$

Remark 2.13. One can also define Brown–Gitler comodules using the same weight filtration on the subalgebras $(\mathcal{A} /\!/ \mathcal{E}(n))^\vee$, leading to a very different family of comodules. As we will only be working with the case defined above, there should be no ambiguity in the notation.

We will specialize to the following two cases throughout our work.

Definition 2.14. (compare with [CQ21, Def. 3.10]) The *k-th integral motivic Brown–Gitler comodule* is

$$B_0^F(k) := \langle x \in (\mathcal{A} /\!/ \mathcal{A}(0))_F^\vee : wt(x) \leq 2k \rangle.$$

The *k-th kq motivic Brown–Gitler comodule* is

$$B_1^F(k) := \langle x \in (\mathcal{A} /\!/ \mathcal{A}(1))_F^\vee : wt(x) \leq 4k \rangle.$$

We remark that in [CQ21], different notation is used for the Brown–Gitler comodules.

Example 2.15. The motivic Brown–Gitler comodule $B_0^F(0)$ is given by

$$B_0^F(0) = \langle x \in (\mathcal{A} /\!/ \mathcal{A}(0))_F^\vee : wt(x) \leq 0 \rangle \cong \mathbb{M}_2^F,$$

with trivial $\mathcal{A}(1)_F^\vee$-comodule structure.

Example 2.16. The motivic Brown–Gitler comodule $B_0^F(1)$ is given by

$$B_0^F(1) = \langle x \in (\mathcal{A} /\!/ \mathcal{A}(0))_F^\vee : wt(x) \leq 2 \rangle = \mathbb{M}_2^F\{1, \bar{\xi}_1, \bar{\tau}_1\},$$

where the $\mathcal{A}(1)_F^\vee$-comodule structure is given by

$$\psi(\bar{\xi}_1) = 1 \otimes \bar{\xi}_1 + \bar{\xi}_1 \otimes 1, \quad \psi(\bar{\tau}_1) = 1 \otimes \bar{\tau}_1 + \bar{\tau}_0 \otimes \bar{\xi}_1.$$

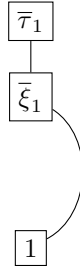In other words, $B_0^F(1)$ is presented as the algebraic cell complex in Fig. 1.

FIGURE 1. The motivic Brown–Gitler comodule $B_0^F(1)$

In fact, we can realize $B_0^F(1)$ as the mod-2 motivic cohomology of a spectrum [CQ21, Example 3.12]. There is a geometric classifying space [MV99, Section 4] $B\mu_2$ of the affine group scheme μ_2 of second roots of unity. Take L_2 to be the simplicial 4-skeleton of $B\mu_2$. The inclusion of the $(3,2)$ cell of L_2 gives a map

$$S^{3,2} \hookrightarrow L_2.$$

Let X be the cofiber; then $\mathrm{H}\mathbb{Z}_1^F := \Sigma^{4,2}F(X, \mathbb{S})$ is a finite complex realizing $B_0^F(1)$ in homology.

Remark 2.17. In classical topology there exist integral and bo Brown–Gitler spectra [Shi83; GJM86]. These are finite spectra $\mathrm{H}\mathbb{Z}_k$ and bo_k, respectively, with the property that

$$\mathrm{H}_*(\mathrm{H}\mathbb{Z}_k) \cong B_0^{cl}(k), \quad \mathrm{H}_*(\mathrm{bo}_k) \cong B_1^{cl}(k).$$

However, while C_2-equivariant analogues exist for integral Brown–Gitler spectra [LPT24], there are currently no constructions of any motivic Brown–Gitler spectra. One possible construction could come from the motivic lambda algebra of Balderamma–Culver–Quigley [BCQ23] and using the classical work of Goerss–Jones–Mahowald [GJM86] as inspiration. An alternative construction, at least over $\mathbb{C}$, is in using the theory of filtered spectra in [Ghe+22]. We will explore the construction of motivic Brown-Gitler spectra in future work.

The $\mathcal{A}(1)_F^\vee$-coaction on $(\mathcal{A} /\!/ \mathcal{A}(1))_F^\vee$ preserves the Mahowald weight, as can be seen by our formulae in Proposition 2.11. This gives the following:

Proposition 2.18. *The motivic integral Brown–Gitler comodules* $B_0^F(k)$ *are* $\mathcal{A}(1)_F^\vee$-*subcomodules of* $(\mathcal{A} /\!/ \mathcal{A}(0))_F^\vee$.

A key property of the integral motivic Brown–Gitler comodules is that they give rise to a splitting of the homology $\mathrm{H}_{**}(\mathrm{kq})$. We refer the reader to [CQ21] for a proof.

Theorem 2.19. *[CQ21, Theorem 3.20] There is an isomorphism of* $\mathcal{A}(1)_F^\vee$-*comodules*

$$(\mathcal{A} /\!/ \mathcal{A}(1))_F^\vee \cong \bigoplus_{k \geq 0} \Sigma^{4k,2k} B_0^F(k).$$

The following result will also be useful.

Proposition 2.20. *There is an isomorphism of* $\mathcal{A}(1)_F^\vee$-*comodules:*

$$\mathrm{H}_{**}(\overline{\mathrm{kq}}) \cong \bigoplus_{k \geq 1} \Sigma^{4k,2k} B_0^F(k).$$

Proof. The defining cofiber sequence

$$\mathbb{S} \to \mathrm{kq} \to \overline{\mathrm{kq}}$$

gives a long exact sequence in homology of the form

$$\cdots \to \mathrm{H}_{**}(\mathbb{S}) \to \mathrm{H}_{**}(\mathrm{kq}) \to \mathrm{H}_{**}(\overline{\mathrm{kq}}) \to \mathrm{H}_{*,*-1}(\mathbb{S}) \to \cdots.$$

We have isomorphisms $\mathrm{H}_{**}(\mathbb{S}) \cong \mathbb{M}_2^F \cong B_0^F(0)$ and $\mathrm{H}_{**}(\mathrm{kq}) \cong \bigoplus_{k \geq 0} \Sigma^{4k,2k} B_0^F(k)$. The unit map induces an inclusion of the bottom summand $B_0^F(0) \hookrightarrow \bigoplus_{k \geq 0} B_0^F(k)$, so by the long exact sequence, $\mathrm{H}_{**}(\overline{\mathrm{kq}})$ is trivial in those degrees. The long exact sequence also gives an isomorphism between $\mathrm{H}_{**}(\mathrm{kq})$ and $\mathrm{H}_{**}(\overline{\mathrm{kq}})$ outside of these degrees since $\mathrm{H}_{**}(\mathbb{S})$ vanishes, giving the result. $\square$

We close this section by recalling useful short exact sequences of $\mathcal{A}(1)_F^\vee$-comodules relating motivic Brown–Gitler comodules. We refer the reader to [CQ21] for a proof.

Proposition 2.21. *[CQ21, Lemma 3.21] There are short exact sequences of* $\mathcal{A}(1)_F^\vee$-*comodules:*

$$0 \to \Sigma^{4k,2k} B_0^F(k) \to B_0^F(2k) \to B_1^F(k-1) \otimes_{\mathbb{M}_2^F} (\mathcal{A}(1) /\!/ \mathcal{A}(0))_F^\vee \to 0,$$

$$0 \to \Sigma^{4k,2k} B_0^F(k) \otimes_{\mathbb{M}_2^F} B_0^F(1) \to B_0^F(2k+1) \to B_1^F(k-1) \otimes_{\mathbb{M}_2^F} (\mathcal{A}(1) /\!/ \mathcal{A}(0))_F^\vee \to 0.$$

Remark 2.22. The short exact sequences in Proposition 2.21 are motivic analogues of classical short exact sequences of Brown–Gitler comodules. These arise by applying mod-2 homology to cofiber sequences relating integral and bo Brown–Gitler spectra [Pea14]. One would imagine the same to be true for the conjectural motivic Brown–Gitler spectra.

3. A SPLITTING OF SYMPLECTIC K-THEORY

In this section, we record a motivic analogue of a classical splitting result. Let $\mathrm{bsp} \simeq \tau_{\geq 0}\Sigma^4 \mathrm{KO}$ denote the connective cover of $\Sigma^4 \mathrm{KO}$. Then there is an equivalence of spectra [MM76, Theorem 1.5]:

$$\mathrm{bsp} \simeq \mathrm{bo} \otimes \mathrm{H}\mathbb{Z}_1,$$

where $\mathrm{H}\mathbb{Z}_1$ denotes the first integral Brown–Gitler spectrum. We prove our result by using the very effective slice filtration.

3.1. **Very effective slice filtration.** The very effective slice filtration was introduced by Spitzweck–Østvær in [SØ12, Section 5] and further studied by Bachmann in [Bac17]. We follow the treatments given in [Ban25, Section 1] and [BH21b, Section 13].

The category of *very effective motivic spectra*, denoted $\mathrm{SH}(F)^{\mathrm{veff}}$, is the full subcategory of $\mathrm{SH}(F)$ generated under colimits by $\Sigma^{n,0}\Sigma_+^\infty X$, where $X \in \mathrm{Sm}_F$ and $n \geq 0$. The category of *very n-effective motivic spectra* for $n \in \mathbb{Z}$, denoted $\mathrm{SH}(F)^{\mathrm{veff}}(n)$, has objects $\Sigma^{2n,n}X$ for $X \in \mathrm{SH}(F)^{\mathrm{veff}}$. These categories assemble into colimit-preserving inclusions

$$\cdots \subset \mathrm{SH}(F)^{\mathrm{veff}}(n+1) \subset \mathrm{SH}(F)^{\mathrm{veff}}(n) \subset \mathrm{SH}(F)^{\mathrm{veff}}(n-1) \subset \cdots \mathrm{SH}(F).$$

In particular, the inclusion functor $\tilde{\mathrm{i}}_n : \mathrm{SH}(F)^{\mathrm{veff}}(n) \to \mathrm{SH}(F)$ admits a right adjoint called the *n-th very effective cover*, denoted $\tilde{\mathrm{f}}_n$. If $X \in \mathrm{SH}(F)$ is any motivic spectrum, the very effective covers give a functorial filtration

$$\cdots \to \tilde{\mathrm{f}}_{n+1}X \to \tilde{\mathrm{f}}_n X \to \tilde{\mathrm{f}}_{n-1}X \to \cdots \to X$$

which we will call the *very effective slice tower* for X.

Example 3.1. In this notation, we have that $\mathrm{kq} = \tilde{\mathrm{f}}_0 \mathrm{KQ}$.

The *n-th very effective slice* of a motivic spectrum X, denoted $\tilde{\mathrm{s}}_n X$, is defined by the cofiber sequence

$$\tilde{\mathrm{f}}_{n+1}X \to \tilde{\mathrm{f}}_n X \to \tilde{\mathrm{s}}_n X.$$

By construction, the very effective cover and slice functors behave well with respect to $\mathbb{P}^1$-suspension.

Lemma 3.2. *[Bac17, Lemma 8] For any $X \in \mathrm{SH}(F)$, we have*

$$\tilde{\mathrm{f}}_{n+1}(\Sigma^{2,1}X) \simeq \Sigma^{2,1}\tilde{\mathrm{f}}_n X, \quad \tilde{\mathrm{s}}_{n+1}(\Sigma^{2,1}X) \simeq \Sigma^{2,1}\tilde{\mathrm{s}}_n X.$$

Remark 3.3. Heuristically, the very effective slice filtration acts as a motivic stand in for the Whitehead tower in classical stable homotopy. We give an example to justify this comparison. Let A and Y be classical spectra, and let $P_n Y$ be defined by the cofiber sequence

$$\tau_{\geq n+1}Y \to \tau_{\geq n}Y \to P_n Y$$

One way to obtain the Atiyah–Hirzebruch spectral sequence

$$E_2 = A_*(P_n Y) \implies A_*(Y)$$

is by smashing the Whitehead tower for Y with A and applying homotopy groups. Now, let E and X be motivic spectra. Then there is a (generalized) very effective slice spectral sequence [CHR24]

$$E_2 = E_{**}(\tilde{\mathrm{s}}_n X) \implies E_{**}(X)$$

obtained by smashing the very effective slice tower for X with E and applying homotopy groups. Over $F = \mathbb{C}$, Betti realization sends the very effective slice spectral sequence to an Atiyah–Hirzebruch spectral sequence.

3.2. **A splitting of symplectic K-theory.** Let ksp denote the very effective cover $\tilde{f}_0(\Sigma^{4,2}KQ)$. Note that by Lemma 3.2, this implies that $\Sigma^{4,2}ksp \simeq \tilde{f}_2KQ$.

Proposition 3.4. *There is an isomorphism of $\mathcal{A}_F^\vee$-comodules:*

$$H_{**}(ksp) \cong (\mathcal{A}\,/\!/\,\mathcal{A}\langle Sq^1, Sq^5\rangle)_F^\vee$$

Proof. The idea is to mimic the classical proof, replacing the Whitehead tower with the very effective slice tower for KQ. This tower gives a cofiber sequence

(3.5) $$\tilde{f}_3KQ \to \tilde{f}_2KQ \to \tilde{s}_2KQ.$$

By definition, we have $\tilde{f}_2KQ \simeq \Sigma^{4,2}ksp$. Bachmann's identification of the very effective slices of KQ [Bac17, Theorem 16] shows that

$$\tilde{s}_2KQ \simeq \Sigma^{4,2}H\mathbb{Z}, \quad \tilde{s}_3KQ \simeq 0.$$

By Bott periodicity for KQ and Lemma 3.2, we have

$$\tilde{f}_4KQ \simeq \tilde{f}_4(\Sigma^{8,4}KQ) \simeq \Sigma^{8,4}\tilde{f}_0KQ = \Sigma^{8,4}kq.$$

Since $\tilde{s}_3KQ \simeq 0$, the defining cofiber sequence gives an equivalence $\tilde{f}_4KQ \simeq \tilde{f}_3KQ \simeq \Sigma^{8,4}kq$. This allows us to rewrite the cofiber sequence (3.5) as

(3.6) $$\Sigma^{8,4}kq \to \Sigma^{4,2}ksp \to \Sigma^{4,2}H\mathbb{Z}.$$

Using the long exact sequence in homology and the isomorphisms from Theorem 2.10

$$H_{**}(H\mathbb{Z}) \cong (\mathcal{A}\,/\!/\,\mathcal{A}(0))_F^\vee, \quad H_{**}(kq) \cong (\mathcal{A}\,/\!/\,\mathcal{A}(1))_F^\vee,$$

and comparing with the short exact sequence of $\mathcal{A}_F^\vee$-comodules:

$$0 \to \Sigma^{4,2}(\mathcal{A}\,/\!/\,\mathcal{A}\langle Sq^1, Sq^5\rangle)_F^\vee \to \Sigma^{4,2}(\mathcal{A}\,/\!/\,\mathcal{A}(0))_F^\vee \to \Sigma^{8,4}(\mathcal{A}\,/\!/\,\mathcal{A}(1))_F^\vee \to 0$$

gives the result, as in the classical case. $\qquad\square$

Recall the integral motivic Brown–Gitler spectrum $H\mathbb{Z}_1^F$ constructed in Example 2.16.

Proposition 3.7. *There is an equivalence of motivic spectra*

$$ksp \simeq kq \otimes H\mathbb{Z}_1^F.$$

Proof. As kq is a ring spectrum, there is a multiplication map

$$\mu : kq \otimes kq \to kq.$$

In homology, this map is the usual multiplication in $\mathcal{A}_F^\vee$, where we identify the left side using Proposition 2.12. This induces the kq-module structure on ksp, realized as a map

$$\mu' : ksp \otimes kq \to ksp.$$

By Proposition 3.4, both Sq^2 and Sq^1Sq^2 act nontrivially on the degree $(0,0)$ generator of $H_{**}(ksp)$, hence there is an inclusion

$$\iota : H\mathbb{Z}_1^F \to ksp$$

inducing the obvious map in homology. One can see now that the composition

$$H\mathbb{Z}_1^F \otimes kq \xrightarrow{\iota \otimes 1} ksp \otimes kq \xrightarrow{\mu'} ksp$$

is an isomorphism in homology. Since we are working in the 2-complete category, this gives an equivalence, finishing the proof. $\qquad\square$

As we will see, this spectrum level splitting allows us to calculate $\pi_{**}^F ksp$ in a particularly nice way.

Remark 3.8. There is another way to relate the motivic spectra ksp and kq that is relevant to our work. In [BH21a], Bachmann-Hopkins construct Adams operations ψ^q on kq. In [KQ24], the homotopy groups of the spectrum L were computed, where L sits in a cofiber sequence

$$L \to kq \xrightarrow{\psi^3 - 1} kq.$$

However, Bachmann-Hopkins show that the map $\psi^3 - 1$ factors through the second very effective cover of kq. At least 2-locally, this defines a motivic image of J spectrum j_O, which sits in a cofiber sequence

$$j_O \to kq \xrightarrow{\psi^3 - 1} \Sigma^{4,2} ksp.$$

In future work, we will show that the homotopy groups $\pi_{**}^F j_O$ are isomorphic to the 0 and 1-lines of the kq-resolution.

4. The inductive process to calculate $\pi_{**}^F(kq \otimes kq)$

In this section, we consider the motivic Adams spectral sequence converging to $\pi_{**}^F(kq \otimes kq)$. We show that the E_2-page splits using motivic Brown–Gitler comodules, then outline an inductive process to compute the E_2-page by a series of algebraic Atiyah–Hirzebruch spectral sequences.

4.1. The motivic Adams spectral sequence for $\pi_{**}^F(\mathbf{kq} \otimes \mathbf{kq})$.
There is an $H\mathbb{F}_2$-motivic Adams spectral sequence computing the cooperations ($\mathbf{mASS}(kq \otimes kq)$). Since $H\mathbb{F}_2$ is flat in the sense of Adams, this takes the form [HKO11]:

$$E_2^{s,f,w} = \mathrm{Ext}_{\mathcal{A}_F^\vee}^{s,f,w}(H_{**}(kq \otimes kq)) \implies \pi_{s,w}^F(kq \otimes kq), \quad d_r : E_r^{s,f,w} \to E_r^{s-1,f+r,w}.$$

We can rewrite the E_2-page of this spectral sequence in a convenient way.

Theorem 4.1. *The E_2-page of the* $\mathbf{mASS}(kq \otimes kq)$ *is given by*

$$E_2^{s,f,w} = \bigoplus_{k \geq 0} \mathrm{Ext}_{\mathcal{A}(1)_F^\vee}^{s,f,w}(\Sigma^{4k,2k} B_0^F(k)).$$

Proof. By Theorem 2.10 and Proposition 2.12, we have an isomorphism of $\mathcal{A}_F^\vee$-comodule algebras

$$H_{**}(kq \otimes kq) \cong (\mathcal{A} /\!/ \mathcal{A}(1))_F^\vee \otimes_{\mathbb{M}_2^F} (\mathcal{A} /\!/ \mathcal{A}(1))_F^\vee.$$

Combined with a standard change of rings argument [Rav86, Appendix A], we can rewrite the E_2-page of the $\mathbf{mASS}(kq \otimes kq)$ as

$$E_2^{s,f,w} = \mathrm{Ext}_{\mathcal{A}(1)_F^\vee}^{s,f,w}((\mathcal{A} /\!/ \mathcal{A}(1))_F^\vee).$$

Finally, using Theorem 2.19 and the fact that $\mathbb{M}_2^F$ is compact as an $\mathcal{A}(1)_F^\vee$-comodule, we can rewrite this as

$$E_2^{s,f,w} = \bigoplus_{k \geq 0} \mathrm{Ext}_{\mathcal{A}(1)_F^\vee}^{s,f,w}(\Sigma^{4k,2k} B_0^F(k)),$$

proving the theorem. $\qquad\square$

Thus, to compute the E_2-page of the $\mathbf{mASS}(kq \otimes kq)$, we must compute the trigraded groups $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(k))$ for $k \geq 0$. Note that as the integral Brown–Gitler comodules are subcomodules of $(\mathcal{A} /\!/ \mathcal{A}(0))_F^\vee$, we understand their structure as $\mathcal{A}(1)_F^\vee$-comodules.

Remark 4.2. Up until this point, all computation has been a precise motivic analogue of the classical work of Mahowald [Mah81]. However, Mahowald's computation of $\pi_*(bo \otimes bo)$ heavily relies on the fact that the classical integral Brown–Gitler spectra $H\mathbb{Z}_k$ give rise to a topological splitting:

$$bo \otimes bo \simeq \bigoplus_{k \geq 0} \Sigma^{4k} bo \otimes H\mathbb{Z}_k.$$

As we have noted, there are currently no known motivic analogues of integral Brown–Gitler spectra except in special cases, and so we must use different methods.

4.2. **The algebraic Atiyah–Hirzebruch spectral sequence.** We now wish to compute the groups $\text{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(k))$ appearing in the decomposition of the E_2-page from Theorem 4.1. To do this, one can use a series of algebraic Atiyah–Hirzebruch spectral sequences (**aAHSS**) along with the short exact sequences of Brown–Gitler comodules from Proposition 2.21. This is originally inspired by the work of Behrens-Ormsby-Stapleton-Stojanoska [Beh+19]. We explain the method here for clarity:

(1) Compute $\text{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(1))/(v_1\text{-torsion})$ using an **aAHSS**:

$$\text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F) \otimes \mathbb{M}_2^F\{[1], [\bar{\xi}_1], [\bar{\tau}_1]\} \implies \text{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(1));$$

(2) Compute $\text{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(k))/(v_1\text{-torsion})$ by induction and Proposition 2.21.

Remark 4.3. Since we are only concerned with using the kq-resolution to compute the v_1-periodic component of $\pi_{**}^F\mathbb{S}$, passing to v_1-torsion free Ext groups loses no information. Additionally, we will see that there are multiple v_1-torsion classes arising in these spectral sequences, and ignoring these classes drastically simplifies our calculations.

We start with the first step. We have that $B_0^F(0) \cong \mathbb{M}_2^F$, and so the first summand of the **mASS**(kq⊗kq) is given by $\text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F)$. Notice that since $\text{H}_{**}(\text{kq}) \cong (\mathcal{A} /\!\!/ \mathcal{A}(1))_F^\vee$ by Theorem 2.10, a standard change of rings argument shows that the E_2-page of the **mASS**(kq) is given by

$$E_2 = \text{Ext}_{\mathcal{A}_F^\vee}(\text{H}_{**}(\text{kq})) \cong \text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F).$$

There are explicit formulae for this algebra in $\mathbb{C}$-motivic [IS11, Theorem 6.6], $\mathbb{R}$-motivic [Hil11, Theorem 4.8], [Gui+20, Theorem 6.2], and $\mathbb{F}_q$-motivic homotopy [Kyl15, Theorems 4.4.2, 4.4.3]. We can use $\text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F)$ as input for an algebraic Atiyah–Hirzebruch spectral sequence **aAHSS**($B_0^F(1)$) computing the next summand, $\text{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(1))$, which we now outline.

Filter $B_0^F(1) = \mathbb{M}_2^F\{1, \bar{\xi}_1, \bar{\tau}_1\}$ by topological degree. Let $\text{F}_i B_0^F(1)$ denote the subspace spanned by generators of degree $\leq i$. This gives a finite filtration of the form

$$
\begin{array}{ccccccc}
0 \longrightarrow & \text{F}_0 B_0^F(1) & \longrightarrow & \text{F}_1 B_0^F(1) & \longrightarrow & \text{F}_2 B_0^F(1) & \longrightarrow & \text{F}_3 B_0^F(1) = B_0^F(1) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& \mathbb{M}_2^F\{[1]\} & & 0 & & \mathbb{M}_2^F\{[\bar{\xi}_1]\} & & \mathbb{M}_2^F\{[\bar{\tau}_1]\}
\end{array}
$$

We refer to this filtration as the *cellular filtration*. The **aAHSS**($B_0^F(1)$) comes from the exact couple arising from applying the functor $\text{Ext}_{\mathcal{A}(1)_F^\vee}(-)$ to this filtration. This spectral sequence has signature

$$E_1^{s,f,w,a} = \text{Ext}_{\mathcal{A}(1)_F^\vee}^{s,f,w}(\mathbb{M}_2^F) \otimes \mathbb{M}_2^F\{[1], [\bar{\xi}_1], [\bar{\tau}_1]\} \implies \text{Ext}_{\mathcal{A}(1)_F^\vee}^{s,f,w}(B_0^F(1)).$$

The E_1-page of this spectral sequence consists of 3 copies of $\text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F)$ in Atiyah–Hirzebruch filtration degrees 0, 2, and 3, which we index by a. Each Atiyah–Hirzebruch filtration piece is suitably shifted by the degree of the cell it is attached to. To make this precise, we can rewrite the E_1-page as

$$E_1 \cong \text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F) \oplus \Sigma^{2,1}\text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F) \oplus \Sigma^{3,1}\text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F).$$

We give a heuristic for this spectral sequence in Fig. 2.

If $x \in \text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F)$ has degree (s, f, w, a), then $|\Sigma^{p,q}x| = (s + p, f, w + q, a)$. We will let $\alpha[i]$ denote the copy of the class $\alpha \in \text{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F)$ in Atiyah–Hirzebruch filtration i. With this notation, differentials in the **aAHSS**($B_0^F(1)$) are of the form

$$d_r : E_r^{s,f,w,a} \to E_r^{s-1,f+1,w,a-r}.$$

Thus the d_r differential lowers Atiyah–Hirzebruch filtration by r. In particular, the cell structure of $B_0^F(1)$ ensures that each differential only takes nonzero value on one Atiyah–Hirzebruch filtration piece.

The differentials in this spectral sequence are computed by the cobar complex: to determine the differential on a class $\alpha[k]$, we first lift to a representative in the cobar complex for $\text{F}_k B_0^F(1)$, compute
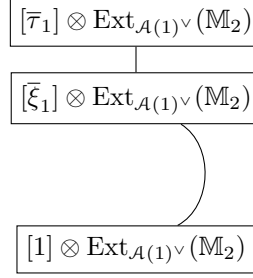
FIGURE 2. A heuristic for the **aAHSS**$(B_0^F(1))$.

the cobar differential on this representative, then project down to the appropriate filtration. In other words, the differentials witness the attaching maps in the algebraic cell complex for $B_0^F(1)$. The d_1-differential witnesses the h_0-attaching map between the cells $[\overline{\tau}_1]$ and $[\overline{\xi}_1]$, so is nonzero only on Atiyah–Hirzebruch filtration 3 and lands in Atiyah–Hirzebruch filtration 2. This gives the following formula:

Proposition 4.4. *In the* **aAHSS**$(B_0^F(1))$, *the d_1-differential is determined by*

$$d_1(\alpha[3]) = h_0\alpha[2].$$

Linearity over $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F)$ combined with the above formula gives the behavior of d_1. The d_2-differential witness the h_1-attaching map between the cells $[\overline{\xi}_1]$ and $[1]$, so is nonzero only on Atiyah–Hirzebruch filtration 2 and lands in Atiyah–Hirzebruch filtration 0. This gives the following formula:

Proposition 4.5. *In the* **aAHSS**$(B_0^F(1))$, *the d_2-differential is determined by*

$$d_2(\alpha[2]) = h_1\alpha[0].$$

Linearity combined with the above formula gives the behavior of d_2. By inspection, the only possible nonzero d_3 differential is between Atiyah–Hirzebruch filtrations 3 and 0, and will be given by the Massey product

$$d_3(\alpha[3]) = \langle \alpha, h_0, h_1 \rangle[0].$$

We will see that this differential is dependent on the base field F, as is evident from the formula. By inspection of the cellular filtration on $B_0^F(1)$, we see the following.

Proposition 4.6. *In the* **aAHSS**$(B_0^F(1))$, *we have that $E_4 = E_\infty$.*

The process of solving extension problems on the E_∞-page is more involved, so we postpone the discussion until we specialize to particular fields F.

Remark 4.7. Though not tantamount to computing the ring of cooperations, it still provides insight to compute $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(1)^{\otimes i})$. Classically, Mahowald is able to express this group in terms of Adams covers of bo and bsp. This is not quite the case here; see Proposition 5.5 and Theorem 7.10.

Now, we discuss the second step of our method. While one could analyze the algebraic Atiyah–Hirzebruch spectral sequence arising from the cellular filtration on $B_0^F(k)$, this spectral sequence becomes drastically more complicated. We will proceed in a different manner. After finding a formula for $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(1))$, one can induct along the short exact sequences of Proposition 2.21 to compute $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(k))$. If k is even, then there is a short exact sequence

$$0 \to \Sigma^{2k,k} B_0^F(\tfrac{k}{2}) \to B_0^F(k) \to B_1^F(\tfrac{k}{2} - 1) \otimes_{\mathbb{M}_2^F} (\mathcal{A}(1) /\!/ \mathcal{A}(0))_F^\vee \to 0.$$

Applying $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(-)$, we see that modulo v_1-torsion the connecting homomorphism is trivial. This reduces the computation of $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(k))$ to the kernel and cokernel in these short exact sequences

in Ext. The kernels are taken care of by the inductive hypothesis, and the cokernels are a simple computation involving a change of rings isomorphism. This finishes the computation when k is even.

When k is odd, there is a short exact sequence

$$0 \to \Sigma^{2(k-1),k-1} B_0^F(\tfrac{k-1}{2}) \otimes_{\mathbb{M}_2^F} B_0^F(1) \to B_0^F(k) \to B_1^F(\tfrac{k-1}{2}-1) \otimes_{\mathbb{M}_2^F} (\mathcal{A}(1) /\!/ \mathcal{A}(0))_F^\vee \to 0.$$

As before, the long exact sequence induced by $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(-)$ splits into short exact sequences. The cokernels are again a simple computation, but the kernels require a more delicate analysis. Indeed, to compute $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}\left(B_0^F(\tfrac{k-1}{2}) \otimes_{\mathbb{M}_2^F} B_0^F(1)\right)$, we use another **aAHSS**. This spectral sequence comes from applying $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}\left(B_0^F(\tfrac{k-1}{2}) \otimes_{\mathbb{M}_2^F} -\right)$ to the previous cellular filtration on $B_0^F(1)$. The E_1-page is given by

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)^\vee}(B_0^F(\tfrac{k-1}{2})) \otimes_{\mathbb{M}_2^F} \mathbb{M}_2^F\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\}.$$

The induction hypothesis handles the left side of the tensor product, and a careful analysis of the spectral sequence gives us the Ext group in question. This finishes the computation when k is odd.

By Theorem 4.1, assembling these results gives the E_2-page of the **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$, modulo v_1-torsion. We can now hope to analyze this spectral sequence by comparing with the classical case and base change. In [CQ21], this was shown to collapse on the E_2-page for all algebraically closed fields of characteristic 0, such as $\mathbb{C}$. We will show that this is also true over $\mathbb{R}$.

Remark 4.8. One can also use a similar analysis to analyze the n-line of the kq-resolution, which takes the form

$$E_1^{*,n,*} = \pi_{*+n,*}^F(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}).$$

We may compute the n-line of the kq-resolution by a motivic Adams spectral sequence. The Künneth spectral sequence for $\mathrm{H}_{**}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ takes the form

$$E_2 = \mathrm{Tor}^{\mathbb{M}_2^F}(\mathrm{H}_{**}(\mathrm{kq}), \mathrm{H}_{**}(\overline{\mathrm{kq}}^{\otimes n})) \implies \mathrm{H}_{**}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}).$$

Using Proposition 2.20, one sees that this collapses on the E_2-page. This implies that

$$\mathrm{H}_{**}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}) \cong \mathrm{H}_{**}(\mathrm{kq}) \otimes_{\mathbb{M}_2^F} \mathrm{H}_{**}(\overline{\mathrm{kq}})^{\otimes n}.$$

After using the change of rings isomorphism on $\mathrm{H}_{**}(\mathrm{kq})$, the **mASS**$(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ takes the form

$$E_2 = \bigoplus_{I \in \mathcal{I}_n} \Sigma^{4|I|,2|I|} \mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(I)) \implies \pi_{*+n,*}^F(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}),$$

where $\mathcal{I}_n = \{I = (i_1, \ldots, i_n) : i_j \geq 0 \text{ for all } 1 \leq j \leq n\}$, and if $I = (i_1, \ldots, i_n)$, then $|I| = i_1 + \cdots + i_n$ and $B_0^F(I) = B_0^F(i_1) \otimes \cdots \otimes B_0^F(i_n)$. Once one has determined the structure of the groups $\mathrm{Ext}_{\mathcal{A}(1)_F^\vee}(B_0^F(k))$ through the process of computing the ring of cooperations, one can study this spectral sequence in detail to compute the E_1-page of the kq-resolution.

5. Computations in SH($\mathbb{C}$)

In this section, we review Culver–Quigley's [CQ21] computation of $\pi_{**}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$. This serves both as an example of the methods described in Section 2 and as a place to recall results which will be useful in the sequel via base change. There are a few places where our arguments differ from the ones given in [CQ21]; we will explicitly indicate when this is the case.

5.1. **Background.** Let $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ denote the mod-2 motivic homology of a point [Voe03a, Corollary 6.10], where $|\tau| = (0, -1)$. Recall that as $\rho = 0$ in $\mathbb{M}_2^{\mathbb{C}}$, the dual Steenrod algebra is an honest Hopf algebra (see Remark 2.9). For the reader's convenience, we review its structure.

Theorem 5.1. *[Voe03b, Theorem 12.6] The dual motivic Steenrod algebra $\mathcal{A}_{\mathbb{C}}^\vee = \pi_{**}^{\mathbb{C}}(\mathrm{H}\mathbb{F}_2 \otimes \mathrm{H}\mathbb{F}_2)$ is the commutative Hopf algebra given by*

$$\mathcal{A}_{\mathbb{C}}^\vee = \mathbb{M}_2^{\mathbb{C}}[\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\tau}_0, \overline{\tau}_1, \ldots]/(\overline{\tau}_i^2 = \tau \overline{\xi}_{i+1}),$$

where $|\overline{\xi}_i| = (2^{i+1} - 2, 2^i - 1)$ and $|\overline{\tau}_i| = (2^{i+1} - 1, 2^i - 1)$. The unit map is given by inclusion, and the coproduct is determined by the formulae:

$$\Delta(\overline{\xi}_k) = \sum_{i+j=k} \overline{\xi}_i \otimes \overline{\xi}_j^{2^i}, \quad \Delta(\overline{\tau}_i) = \sum_{i+j=k} \overline{\tau}_i \otimes \overline{\xi}_j^{2^i} + 1 \otimes \overline{\tau}_k.$$

Remark 5.2. There is a well understood relationship between $\mathcal{A}_\mathbb{C}^\vee$ and the classical dual Steenrod algebra $\mathcal{A}^\vee$. Complex Betti realization gives a map $\mathrm{Be}^\mathbb{C} : \mathcal{A}_\mathbb{C}^\vee \to \mathcal{A}^\vee$ determined by

$$\mathrm{Be}^\mathbb{C}(\tau) = 1, \quad \mathrm{Be}^\mathbb{C}(\overline{\xi}_i) = \overline{\xi}_i^2, \quad \mathrm{Be}^\mathbb{C}(\overline{\tau}_i) = \overline{\xi}_i$$

This extends to an isomorphism in Ext [DI10]:

$$\mathrm{Ext}_{\mathcal{A}_\mathbb{C}^\vee}(\mathbb{M}_2^\mathbb{C}) \otimes_{\mathbb{M}_2^\mathbb{C}} \mathbb{M}_2^\mathbb{C}[\tau^{-1}] \cong \mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2) \otimes \mathbb{F}_2[\tau^{\pm 1}].$$

This observation is at the heart of the theory of synthetic spectra [Pst23], and will allow us to determine motivic information from classical information.

In addition to an understanding of the homology $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} /\!/ \mathcal{A}(1))_\mathbb{C}^\vee$, we also have a computation of the homotopy groups $\pi_{**}^\mathbb{C}(\mathrm{kq})$ [IS11].

Theorem 5.3. *[IS11, Theorem 4.10]* *The* **mASS**(kq) *has* E_2*-page*

$$E_2^{s,f,w} = \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(\mathbb{M}_2^\mathbb{C}) = \frac{\mathbb{M}_2^\mathbb{C}[h_0, h_1, a, b]}{(h_0 h_1, \tau h_1^3, h_1 a, a^2 = h_0^2 b)},$$

where $|h_0| = (0, 1, 0)$, $|h_1| = (1, 1, 1)$, $|a| = (4, 3, 2)$ *and* $|b| = (8, 4, 4)$. *Moreover, the spectral sequence collapses at* E_2 *with no extensions, hence there is an isomorphism*

$$\pi_{**}^\mathbb{C} \mathrm{kq} \cong \frac{\mathbb{Z}_2^\wedge[\tau, h_1, a, b]}{(2h_1, \tau h_1^3, h_1 a, a^2 = 4b)},$$

where the stem and filtration of the generators match those above.

Following our discussion in Section 2, the first step towards computing the kq-resolution is computing the ring of cooperations $\pi_{**}^\mathbb{C}(\mathrm{kq} \otimes \mathrm{kq})$. Recall from Theorem 4.1 that the **mASS**(kq $\otimes$ kq) has E_2-page given by

$$E_2^{s,f,w} = \bigoplus_{k \geq 0} \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}^{s,f,w}(\Sigma^{4k,2k} B_0^\mathbb{C}(k)),$$

where $B_0^\mathbb{C}(k)$ are the $\mathbb{C}$-motivic integral motivic Brown–Gitler comodules. Therefore, we must compute the trigraded groups $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(B_0^\mathbb{C}(k))$ for $k \geq 0$. Since $B_0^\mathbb{C}(0) = \mathbb{M}_2^\mathbb{C}$, the first summand is computed by Theorem 5.3. We move then to compute $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(B_0^\mathbb{C}(1))$ by the algebraic Atiyah–Hirzebruch spectral sequence outlined in Section 4.2.

5.2. $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(B_0^\mathbb{C}(1)^{\otimes i})$. Recall that the **aAHSS**($B_0^\mathbb{C}(1)$) takes the form

$$E_1^{s,f,w,a} = \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}^{s,f,w}(\mathbb{M}_2^\mathbb{C}) \otimes_{\mathbb{M}_2^\mathbb{C}} \mathbb{M}_2^\mathbb{C}\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\} \implies \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}^{s,f,w}(B_0^\mathbb{C}(1)),$$

with differentials $d_r : E_r^{s,f,w,a} \to E_r^{s-1,f+1,w,a-r}$. We will let $\alpha[i]$ denote the copy of a class $\alpha \in \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(\mathbb{M}_2^\mathbb{C})$ in Atiyah–Hirzebruch filtration i. We have discussed in Section 4.2 the differentials for this spectral sequence. Namely, the d_1-differential is only nonzero on Atiyah–Hirzebruch filtration 3, where

$$d_1(\alpha[3]) = h_0 \alpha[2],$$

and the d_2-differential is only nonzero on Atiyah–Hirzebruch filtration 2, where

$$d_2(\alpha[2]) = h_1 \alpha[0].$$

There is a potential nonzero third differential $d_3 : E_3^{s,f,w,3} \to E_3^{s-1,f+1,w,0}$. However, for degree reasons we have that $\langle \alpha, h_0, h_1 \rangle = 0$ for all remaining classes $\alpha[3] \in E_3^{s,f,w,3}$, hence this differential is 0. Thus the spectral sequence collapses on $E_3 = E_\infty$.

We depict the E_1-page in Fig. 3, the E_2-page in Fig. 4, and the $E_3 = E_\infty$-page in Fig. 5. Charts are written in (s, f)-grading, with motivic weight and Atiyah–Hirzebruch filtration suppressed. A black ● represents $\mathbb{M}_2^\mathbb{C}$. A hollow ○ represents $\mathbb{F}_2$. A black vertical line represents multiplication by h_0. A diagonal line represents multiplication by h_1. Differentials are blue, linear with respect to the underlying $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(\mathbb{M}_2^\mathbb{C})$-module structure, and preserve motivic weight. A dashed blue line represents a differential where either source or target is τ-torsion.
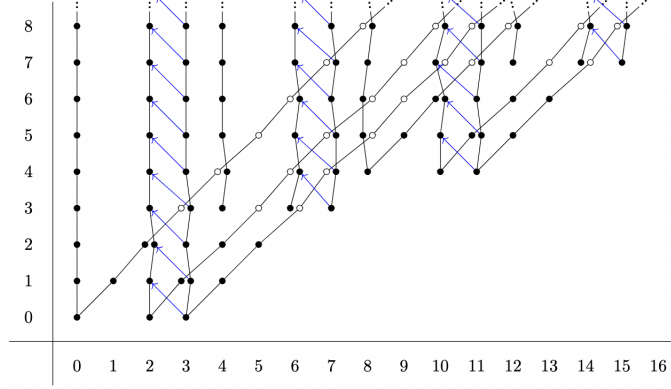

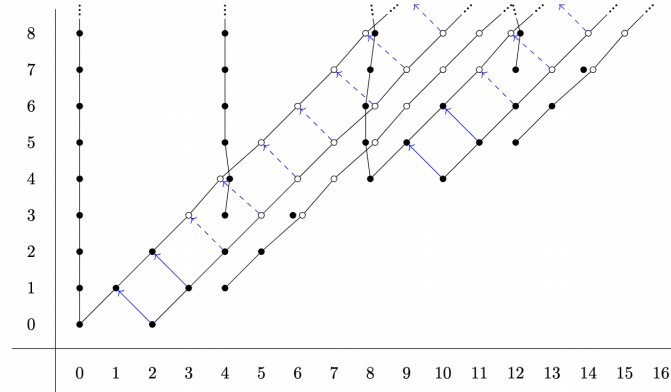
FIGURE 3. The E_1-page of the **aAHSS**$(B_0^\mathbb{C}(1))$

FIGURE 4. The E_2-page of the **aAHSS**$(B_0^\mathbb{C}(1))$

The hidden extensions on the E_∞-page can be resolved using complex Betti realization. Recall from Section 2.1 that this is a functor $\mathrm{Be}^\mathbb{C} : \mathrm{SH}(\mathbb{C}) \to \mathrm{Sp}$. For $X \in \mathrm{SH}(\mathbb{C})$, this gives a map in homotopy
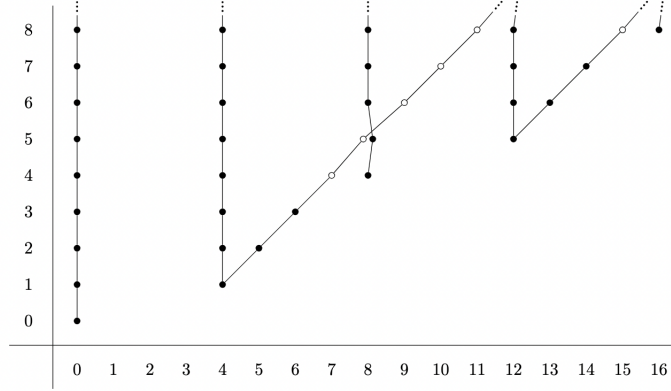
$$\pi_{s,w}^\mathbb{C} X \to \pi_s \mathrm{Be}^\mathbb{C} X.$$

In particular, there is a map of dual Steenrod algebras $\mathcal{A}_\mathbb{C}^\vee \to \mathcal{A}^\vee$ sending $\bar{\xi}_i \mapsto \bar{\xi}_i^2$ and $\bar{\tau}_i \mapsto \bar{\xi}_i$. Additionally, there is an isomorphism of $\mathcal{A}(1)_\mathbb{C}^\vee$-comodules

$$(B_0^\mathbb{C}(1))[\tau^{-1}] \cong B_0(1)[\tau^{\pm 1}],$$

where $B_0(1)$ is the classical Brown–Gitler comodule. This induces a map of Ext groups

$$\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}^{s,f,w}(B_0^\mathbb{C}(k)) \to \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f}(B_0(k)),$$

FIGURE 5. The E_∞-page of the **aAHSS**$(B_0^\mathbb{C}(1))$ with hidden extensions.

and allows one to lift hidden extensions from the classical case in [Mah81] to the $\mathbb{C}$-motivic case.

The following is a motivic analogue of a classical result.

Corollary 5.4. *[CQ21, Corollary 3.31] There is an isomorphism:*

$$\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(\mathbb{M}_2^\mathbb{C}, B_0^\mathbb{C}(1)) \cong \pi_{**}^\mathbb{C}(\mathrm{ksp}).$$

Proof. By Proposition 3.7, there is an equivalence of motivic spectra

$$\mathrm{ksp} \cong \mathrm{H}\mathbb{Z}_1^\mathbb{C} \otimes \mathrm{kq},$$

where $\mathrm{H}\mathbb{Z}_1^\mathbb{C}$ is the motivic spectrum constructed in Section 2.3. The Künneth spectral sequence for mod-2 homology collapses for $\mathrm{H}\mathbb{Z}_1^\mathbb{C} \otimes \mathrm{kq}$, giving an isomorphism

$$\mathrm{H}_{**}(\mathrm{ksp}) \cong B_0^\mathbb{C}(1) \otimes_{\mathbb{M}_2^\mathbb{C}} (\mathcal{A} /\!/ \mathcal{A}(1))_\mathbb{C}^\vee.$$

After a change of rings isomorphism, the E_2-page for the **mASS**(ksp) takes the form

$$E_2 = \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(B_0^\mathbb{C}(1)) \implies \pi_{**}^\mathbb{C}(\mathrm{ksp}).$$

This Ext group was computed by the **aAHSS**$(B_0^\mathbb{C}(1))$ and is depicted in Fig. 5. There are no differentials for degree reasons, so the spectral sequence collapses and gives the result. $\square$

Classically, one can express the groups $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(B_0(1)^{\otimes i})$ in terms of Adams covers of bo and bsp. This does not translate well to motivic homotopy theory. Over $\mathbb{C}$, Culver–Quigley [CQ21] show that one cannot express the groups $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(B_0^\mathbb{C}(1)^{\otimes i})$ in terms of Adams covers of kq and ksp. We show a similar result over $\mathbb{R}$ in Section 7. Instead, there is a periodic family of groups which will determine $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(B_0^\mathbb{C}(1))$ composed of Adams covers of bo, bsp, kq and ksp.

In what follows, we will use the notation defined in Section 1.5. For $i \geq 0$, let $Z_i^\mathbb{C}$ be the trigraded group defined as follows:

- When $i \equiv 0\,(4)$, let

$$Z_i^\mathbb{C} = \bigoplus_{j=0}^{i/2-1} \Sigma^{4j,2j}\mathbb{M}_2^\mathbb{C}[h_0] \oplus \Sigma^{2i,i}\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(\mathbb{M}_2^\mathbb{C})$$

- When $i \equiv 1\,(4)$, let

$$Z_i^\mathbb{C} = \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j,2j}\mathbb{M}_2^\mathbb{C}[h_0] \oplus \Sigma^{2i-1,i-1}\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}(B_0^\mathbb{C}(1));$$

- When $i \equiv 2 \, (4)$, let

$$Z_i^{\mathbb{C}} = \bigoplus_{j=0}^{i/2-1} \Sigma^{4j,2j} \mathbb{M}_2^{\mathbb{C}}[h_0] \oplus \Sigma^{2i-2,i} \mathbb{M}_2^{\mathbb{C}} \oplus \Sigma^{2i,i} \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(1))\langle 1 \rangle;$$

- When $i \equiv 3 \, (4)$, let

$$Z_i^{\mathbb{C}} = \bigoplus_{j=0}^{(i-1)/2} \Sigma^{4j,2j} \mathbb{M}_2^{\mathbb{C}}[h_0] \oplus \Sigma^{2i-1,i} \mathbb{M}_2^{\mathbb{C}}[h_1]/(h_1^2) \oplus \Sigma^{2i+2,i+1} \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}^{***}(B_0^{\mathbb{C}}(1))\langle 2 \rangle.$$

Recall that we are only interested in information up to v_1-torsion (see Remark 4.3).

Proposition 5.5. *[CQ21, Lemma 3.36] There is an isomorphism of $\mathcal{A}(1)_{\mathbb{C}}^{\vee}$-comodules and $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})$-modules:*

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(1)^{\otimes i})}{v_1\text{-torsion}} \cong Z_i^{\mathbb{C}}.$$

Proof. By applying the functor $\mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}\left(B_0^{\mathbb{C}}(1)^{\otimes i} \otimes_{\mathbb{M}_2^{\mathbb{C}}} -\right)$ to the cellular filtration of $B_0^{\mathbb{C}}(1)$, we may inductively compute these Ext groups by an **aAHSS**, where the base case was computed in Corollary 5.4. This spectral sequence takes the form

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(1)^{\otimes i}) \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathbb{M}_2^{\mathbb{C}}\{1, \bar{\xi}_1, \bar{\tau}_1\} \implies \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(1)^{\otimes i+1}).$$

The differentials take the same form as those in **aAHSS**$(B_0^{\mathbb{C}}(1))$, namely we have $d_1(\alpha[3]) = h_0\alpha[2]$ and $d_2(\alpha[2]) = h_1\alpha[0]$, where $\alpha \in \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(1)^{\otimes i})$. For degree reasons, there is no d_3-differential. Hidden extensions may be obtained by comparison with the classical case, using the algebra map

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(1)^{\otimes i}) \to \mathrm{Ext}_{\mathcal{A}(1)^{\vee}}(B_0(1)^{\otimes i}).$$

This resolves all hidden extensions and concludes the proof. $\qquad\square$

5.3. $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(k))$ and the ring of cooperations.

Next, we compute $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(k))$ for all $k \geq 1$.

Theorem 5.6. *[CQ21, Theorem 3.38] There is an isomorphism of $\mathcal{A}(1)_{\mathbb{C}}^{\vee}$-comodules and $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})$-modules:*

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(k))}{v_1\text{-torsion}} \cong \Sigma^{4k-4,2k-2} Z_{\alpha(k)}^{\mathbb{C}} \oplus \bigoplus_{j=0}^{4k-8} \Sigma^{4j,2j} \mathbb{M}_2^{\mathbb{C}}[h_0],$$

where $\alpha(k)$ is the number of 1's in the dyadic expansion of k.

Proof. For notational convenience, all Ext groups in this proof will be implicitly computed modulo v_1-torsion. We will induce on k. The case of $k = 1$ was shown in Corollary 5.4. Now, suppose the theorem is true for all $i < k$.

Suppose that k is even. Proposition 2.21 gives a short exact sequence of $\mathcal{A}(1)_{\mathbb{C}}^{\vee}$-comodules

$$(5.7) \qquad 0 \to \Sigma^{2k,k} B_0^{\mathbb{C}}(\tfrac{k}{2}) \to B_0^{\mathbb{C}}(k) \to B_1^{\mathbb{C}}(\tfrac{k}{2} - 1) \otimes_{\mathbb{M}_2^{\mathbb{C}}} (\mathcal{A}(1) /\!/ \mathcal{A}(0))_{\mathbb{C}}^{\vee} \to 0.$$

Applying the functor $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(-)$ gives a long exact sequence of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})$-modules. Moreover, after killing v_1-torsion the connecting homomorphism is trivial, giving short exact sequences whose middle term is $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(k))$. Therefore, this Ext group decomposes into the Ext groups of the kernel and cokernel of the original short exact sequence (5.7). The kernel is handled by the inductive hypothesis: we have

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\Sigma^{2k,k} B_0^{\mathbb{C}}(\tfrac{k}{2})) \cong \Sigma^{2k,k} \left(\Sigma^{2k-4,k-2} Z_{\alpha(k/2)}^{\mathbb{C}} \oplus \bigoplus_{j=0}^{2k-8} \Sigma^{4j,2j} \mathbb{M}_2^{\mathbb{C}}[h_0] \right).$$

We can use a change of rings isomorphism to rewrite the cokernel as

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}\left(B_1^{\mathbb{C}}(\tfrac{k}{2}-1)\otimes_{\mathbb{M}_2^{\mathbb{C}}}(\mathcal{A}(1)/\!/\mathcal{A}(0))_{\mathbb{C}}^{\vee}\right)\cong\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{C}}^{\vee}}(B_1^{\mathbb{C}}(\tfrac{k}{2}-1)).$$

Since $\mathcal{A}(0)_{\mathbb{C}}^{\vee}\cong\mathbb{M}_2^{\mathbb{C}}[\bar{\tau}_0]/(\bar{\tau}_0^2)$ is exterior, the Ext groups in question are polynomial. Modulo v_1-torsion, we are only left with h_0-towers:

$$\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{C}}^{\vee}}(B_1^{\mathbb{C}}(\tfrac{k}{2}-1))\cong\bigoplus_{j=0}^{k/2-1}\Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0].$$

Putting these pieces together with some reindexing on the h_0-towers and using the fact that $\alpha(k/2)=\alpha(k)$ gives the result.

Suppose now that k is odd. Again, Proposition 2.21 gives a short exact sequence of $\mathcal{A}(1)_{\mathbb{C}}^{\vee}$-comodules

$$(5.8)\quad 0\to\Sigma^{2(k-1),k-1}B_0^{\mathbb{C}}(\tfrac{k-1}{2})\otimes_{\mathbb{M}_2^{\mathbb{C}}}B_0^{\mathbb{C}}(1)\to B_0^{\mathbb{C}}(k)\to B_1^{\mathbb{C}}(\tfrac{k-1}{2}-1)\otimes_{\mathbb{M}_2^{\mathbb{C}}}(\mathcal{A}(1)/\!/\mathcal{A}(0))_{\mathbb{C}}^{\vee}\to 0.$$

Applying the functor $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(-)$ gives a long exact sequence of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})$-modules, and as in the even case, the connecting homomorphism is trivial modulo v_1-torsion. Thus, the Ext group we are interested in decomposes into the Ext groups of the kernel and cokernel of 5.8. In this case, the cokernel is simpler to compute. A change of rings isomorphism gives

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}\left(B_1^{\mathbb{C}}(\tfrac{k-1}{2}-1)\otimes_{\mathbb{M}_2^{\mathbb{C}}}(\mathcal{A}(1)/\!/\mathcal{A}(0))_{\mathbb{C}}^{\vee}\right)\cong\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{C}}^{\vee}}(B_1^{\mathbb{C}}(\tfrac{k-1}{2}-1)).$$

By the same argument as before, modulo v_1-torsion this Ext group consists of purely h_0-towers:

$$\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{C}}^{\vee}}(B_1^{\mathbb{C}}(\tfrac{k-1}{2}-1))\cong\bigoplus_{j=0}^{(k-1)/2-1}\Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0].$$

The Ext group of the kernel may be computed using an algebraic Atiyah–Hirzebruch spectral sequence $\mathbf{aAHSS}\left(B_0^{\mathbb{C}}(\tfrac{k-1}{2})\otimes_{\mathbb{M}_2^{\mathbb{C}}}B_0^{\mathbb{C}}(1)\right)$. This has signature

$$E_1=\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(\tfrac{k-1}{2}))\otimes_{\mathbb{M}_2^{\mathbb{C}}}\mathbb{M}_2^{\mathbb{C}}\{1,\bar{\xi}_1,\bar{\tau}_1\}\implies\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}\left(B_0^{\mathbb{C}}(\tfrac{k-1}{2})\otimes_{\mathbb{M}_2^{\mathbb{C}}}B_0^{\mathbb{C}}(1)\right).$$

By induction, we may rewrite $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(\tfrac{k-1}{2}))$ as

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(\tfrac{k-1}{2}))\cong\Sigma^{2k-6,k-3}Z_{\alpha(k)-1}^{\mathbb{C}}\oplus\bigoplus_{j=0}^{2k-10}\Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0],$$

using that $\alpha(\tfrac{k-1}{2})=\alpha(k)-1$. This splitting leads to a splitting of the E_1-page of the algebraic Atiyah–Hirzebruch spectral sequence, and we may analyze each summand individually. On the left-hand summand, Proposition 5.5 implies that result of the spectral sequence is isomorphic to $\Sigma^{2k-2,k-1}Z_{\alpha(k)}^{\mathbb{C}}$. On the right-hand summand, the spectral sequence collapses on the E_1-page, giving back the original summand modulo v_1-torsion. Thus, after reindexing the h_0-towers, the Ext groups of the kernel and cokernel and right-hand side of the Ext groups assemble to give the result. □

Remark 5.9. The above proof is where our argument differs most from the one given in [CQ21]. We explain these differences here. Classically, Mahowald [Mah81] showed that the groups $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}(B_0(k))$ can be expressed solely in terms of bo and bsp. To be precise, for a spectrum X, let $X^{\langle n\rangle}$ denote the n-th Adams cover of X, so that

$$X\leftarrow X^{\langle 1\rangle}\leftarrow X^{\langle 2\rangle}\leftarrow X^{\langle 3\rangle}\leftarrow\cdots$$

is a minimal Adams resolution of X. Then there are isomorphisms:

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}(B_0(k))}{v_1\text{-torsion}}\cong\begin{cases}\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}(\mathrm{H}_{**}(\mathrm{bo}^{\langle 2k-\alpha(k)\rangle})) & k\equiv 0\,(2),\\ \mathrm{Ext}_{\mathcal{A}(1)^{\vee}}(\mathrm{H}_{**}(\mathrm{bsp}^{\langle 2k-\alpha(k)-1\rangle})) & k\equiv 1\,(2).\end{cases}$$

One thing to note is that in the classical case, each of the above groups is describable by a direct sum of h_0-towers followed by some shift of $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2)$. In particular, if $x \in \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f}(B_0(k))$ is in stem $s < 4(2k - \alpha(k))$, then x does not support h_1-multiplication.

The results of Culver–Quigley assert that over $\mathbb{C}$ there is an isomorphism

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)^\vee_\mathbb{C}}(B_0^\mathbb{C}(k))}{v_1\text{-torsion}} \cong Z_{2k-\alpha(k)}^\mathbb{C}.$$

However, this is not quite correct. The group $Z_{2k-\alpha(k)}^\mathbb{C}$ has classes in small stems which support h_1-multiplication, while the decomposition given by the short exact sequences of Proposition 2.21 do not allow for this. For example, for $k = 2$ we have $2k - \alpha(k) = 4 - 1 = 3$, hence Culver–Quigley claim that $\mathrm{Ext}_{\mathcal{A}(1)^\vee_\mathbb{C}}(B_0^\mathbb{C}(2))$ is isomorphic to

$$Z_3^\mathbb{C} = \mathbb{M}_2^\mathbb{C}[h_0] \oplus \Sigma^{4,2}\mathbb{M}_2^\mathbb{C}[h_0] \oplus \Sigma^{5,3}\mathbb{M}_2^\mathbb{C}[h_1]/(h_1^2) \oplus \Sigma^{8,4}\mathrm{Ext}_{\mathcal{A}(1)^\vee_\mathbb{C}}(B_0^\mathbb{C}(1))\langle 2 \rangle.$$

We see that there is a class in stem 5 which supports an h_1-multiplication. Alternatively, using the short exact sequence of Proposition 2.21 we see that the group $\mathrm{Ext}_{\mathcal{A}(1)^\vee_\mathbb{C}}(B_0^\mathbb{C}(2))$ is isomorphic to following the direct sum:

$$\Sigma^{8,4}\mathrm{Ext}_{\mathcal{A}(1)^\vee_\mathbb{C}}(B_0^\mathbb{C}(1)) \oplus \mathrm{Ext}_{\mathcal{A}(0)^\vee_\mathbb{C}}(B_1^\mathbb{C}(0)) \cong \Sigma^{8,4}Z_1^\mathbb{C} \oplus \mathbb{M}_2^\mathbb{C}[h_0],$$

which agrees with the description in Proposition 5.5. This decomposition does not admit any h_1-multiplication in stem 5; in fact, there are no classes in stem 5. This type of difference extends to all values of $k \geq 2$, showing that the formula given in [CQ21, Theorem 3.38] is incorrect. However, this observation is actually in line with the rest of the work done in Culver–Quigley and doesn't affect any end results. More precisely:

(1) It was shown in [CQ21, Lemma 3.38] that the groups $\mathrm{Ext}_{\mathcal{A}(1)^\vee_\mathbb{C}}(B_0^\mathbb{C}(1)^{\otimes i})$ are not expressible in terms of Adams covers of kq and ksp. The purpose of the groups $Z_i^\mathbb{C}$ is exactly to account for this failure, mixing in an appropriate trigraded version of $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathrm{H}_{**}(\mathrm{bo}^{\langle i \rangle}))$ and $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathrm{H}_{**}(\mathrm{bsp}^{\langle i \rangle}))$. The difference being observed in the computation of $\mathrm{Ext}_{\mathcal{A}(1)^\vee_\mathbb{C}}(B_0^\mathbb{C}(k))$ can be accounted for by noting that this group *also* cannot be expressed using just $Z_{2k-\alpha(k)}^\mathbb{C}$. Rather, we must express these groups using shifts of $Z_{\alpha(k)}^\mathbb{C}$ and an appropriate trigraded version of the h_0-towers found in the classical groups $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2)$; this last term is precisely what is given by $\mathrm{Ext}_{\mathcal{A}(0)^\vee_\mathbb{C}}(B_1^\mathbb{C}(\frac{k-1}{2}))$ in the case that $k \equiv 0\ (2)$, and by $\mathrm{Ext}_{\mathcal{A}(0)^\vee_\mathbb{C}}(B_1^\mathbb{C}(\frac{k-1}{2}-1))$ in the case that $k \equiv 1\ (2)$.

(2) Apart from this discrepancy, the contents of [CQ21] are unaffected. In particular, we will see that the **mASS**(kq $\otimes$ kq) collapses on the E_2-page by using the same arguments as employed by Culver–Quigley.

The following is now immediate from Theorem 4.1.

Proposition 5.10. *[CQ21, Proposition 3.41] The E_2-page of the **mASS**(kq $\otimes$ kq) is given, modulo v_1-torsion, by*

$$E_2 = \bigoplus_{k \geq 0} \mathrm{Ext}_{\mathcal{A}(1)^\vee_\mathbb{C}}^{s,f,w}(\Sigma^{4k,2k}B_0^\mathbb{C}(k)) \cong \bigoplus_{k \geq 0} \left(\Sigma^{4k,2i}Z_{\alpha(k)}^\mathbb{C} \oplus \bigoplus_{j=0}^{4k-8} \Sigma^{4j,2j}\mathbb{M}_2^\mathbb{C}[h_0] \right).$$

Recall from [Mah81, Theorem 2.9] that the **ASS**(bo $\otimes$ bo) collapses on E_2.

Theorem 5.11. *[CQ21, Cor 3.43] The **mASS**(kq $\otimes$ kq) collapses on the E_2-page.*

Proof. Betti realization sends the **mASS**(kq $\otimes$ kq) to the **ASS**(bo $\otimes$ bo). At the level of E_2-pages, this is a map

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee_\mathbb{C}}^{s,f,w}(\mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq})) \to \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f}(\mathrm{H}_*(\mathrm{bo} \otimes \mathrm{bo})),$$

obtained by inverting τ on the E_2-page of the **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$ and setting $\tau = 1$. Since the **ASS**$(\mathrm{bo} \otimes \mathrm{bo})$ collapses on E_2, we must have that there are no motivic differentials where both source and target are τ-torsion free. However, our description of the E_2-page of the **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$ shows that any τ-torsion class must be h_1-torsion free, hence is some h_1-tower. For degree reasons, there can be no such differentials between these towers, concluding the proof. $\square$

Part 2. **Computations in** $\mathrm{SH}(\mathbb{R})$

6. Algebraic preliminaries and comparisons

In this section, we recall the $\mathbb{R}$-motivic dual Steenrod algebra $\mathcal{A}^{\vee}_{\mathbb{R}}$ and introduce notation for modules to be used throughout. Then, we give a brief overview of the relationship between $\mathbb{C}$-motivic and $\mathbb{R}$-motivic homotopy and compare the respective kq-resolutions.

6.1. **The dual motivic Steenrod algebra and** $\mathbf{Ext}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}_2)$. Let $\mathbb{M}^{\mathbb{R}}_2 = \mathbb{F}_2[\rho, \tau]$ denote the mod-2 motivic homology of a point [Voe03a, Corollary 6.10], where $|\tau| = (0, -1)$ and $|\rho| = (-1, -1)$. Recall from Theorem 2.7 that the dual motivic Steenrod algebra $\mathcal{A}^{\vee}_{\mathbb{R}}$ takes the form

$$\mathcal{A}^{\vee}_{\mathbb{R}} = \mathbb{M}^{\mathbb{R}}_2[\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\tau}_0, \overline{\tau}_1, \ldots]/(\overline{\tau}^2_i = \rho\overline{\tau}_{i+1} + \rho\overline{\tau}_0\overline{\xi}_{i+1} + \tau\overline{\xi}_{i+1}).$$

Theorem 4.1 asserts that the **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$ has E_2-page given by

$$E^{s,f,w}_2 = \bigoplus_{k \geq 0} \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\Sigma^{4k,2k} B^{\mathbb{R}}_0(k)),$$

where $B^{\mathbb{R}}_0(k)$ are the $\mathbb{R}$-motivic integral motivic Brown–Gitler comodules. Therefore, to compute this E_2-page we must compute the trigraded groups $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(B^{\mathbb{R}}_0(k))$ for $k \geq 0$. Since $B^{\mathbb{R}}_0(0) = \mathbb{M}^{\mathbb{R}}_2$, the first summand is isomorphic to the E_2-page of the **mASS**(kq). This was computed by Hill in [Hil11, Theorem 5.6]. We use a presentation adopted by Guillou–Hill–Isaksen–Ravenel [Gui+20, Theorem 6.2].

Theorem 6.1. *[Hil11; Gui+20] The* **mASS**(kq) *collapses on the E_2-page, which is given by*

$$\mathrm{Ext}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}_2, \mathbb{M}^{\mathbb{R}}_2) = \mathbb{F}_2[\tau^4, \rho, h_0, h_1, a, b, \tau^2 h_0, \tau h_1, b, \tau^2 a]/I.$$

The degrees of the generators are listed in Table 1 *and the ideal of relations I is detailed in* Table 2.

We organize Table 1 and Table 2 by coweight modulo 4.

Generator	(s, f, w)	Coweight
ρ	$(-1, 0, -1)$	0
h_0	$(0, 1, 0)$	0
h_1	$(1, 1, 1)$	0
$\tau^2 a$	$(4, 3, 0)$	4
τ^4	$(0, 0, -4)$	4
b	$(8, 4, 4)$	4
τh_1	$(1, 1, 0)$	1
$\tau^2 h_0$	$(0, 1, -2)$	2
a	$(4, 3, 2)$	2

TABLE 1. Generators for $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}_2)$

Relation	(s, f, w)	Coweight
ρh_0	$(-1, 1, -1)$	0
$h_0 h_1$	$(1, 2, 1)$	0
$(\tau^2 h_0)^2 + \tau^4 h_0^2$	$(0, 2, -4)$	4
$\tau^4 h_1^3 + \rho \cdot \tau^2 a$	$(3, 3, -1)$	4
$\tau^2 h_0 \cdot a + h_0 \cdot \tau^2 a$	$(4, 4, 0)$	4
$h_1 \cdot \tau^2 a + \rho^3 b$	$(5, 4, 1)$	4
$a^2 + h_0^2 b$	$(8, 6, 4)$	4
$(\tau^2 a)^2 + \tau^4 h_0^2 b + \rho^2 \tau^4 h_1^2 b$	$(8, 6, 0)$	8
$\rho^2 \cdot \tau h_1$	$(-1, 1, -2)$	1
$h_0 \cdot \tau h_1 + \rho h_1 \cdot \tau h_1$	$(1, 2, 0)$	1
$h_1^2 \cdot \tau h_1$	$(3, 3, 2)$	1
$\tau h_1 \cdot \tau^2 a$	$(5, 4, 0)$	5
$\rho \cdot \tau^2 h_0$	$(-1, 1, -3)$	2
$\rho^3 \cdot a$	$(1, 3, -1)$	2
$\tau^2 h_0 \cdot h_1 + \rho(\tau h_1)^2$	$(1, 2, -1)$	2
$h_1(\tau h_1)^2 + \rho a$	$(3, 3, 1)$	2
$h_1 a$	$(5, 4, 3)$	2
$\tau^2 h_0 \cdot \tau^2 a + \tau^4 h_0 a$	$(4, 4, -2)$	6
$a \cdot \tau^2 a + \tau^2 h_0 \cdot h_0 b$	$(8, 6, 2)$	6
$\tau^2 h_0 \cdot \tau h_1$	$(1, 2, -2)$	3
$(\tau h_1)^3$	$(3, 3, 0)$	3
$\tau h_1 \cdot a$	$(5, 4, 2)$	3

TABLE 2. Relations for $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$

The data comprising $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ is too complicated to display in one chart. To present this material more clearly, we give individual charts for each coweight modulo 4, where this 4-periodicity is induced by the generator τ^4. Notice that there are no generators in coweight 3 modulo 4, and the relations ensure that there are never any classes in this coweight. Charts are written in (s, f)-grading, with motivic weight suppressed. We also suppress infinite ρ-towers, which would extend to the left in the charts, for readability. Fig. 6 depicts the coweight 0 piece, Fig. 7 depicts the coweight 1 piece, and Fig. 8 depicts the coweight 2 piece.

A black ■ represents $\mathbb{F}_2[\tau^4][\rho, h_1]$. A black • represents $\mathbb{F}_2[\tau^4]$. A vertical black line represents multiplication by h_0. A horizontal black line represents multiplication by ρ. A diagonal black line represents multiplication by h_1. A dashed horizontal line indicates that ρ-multiplication hits τ^4 times a generator. For example, in Fig. 6 there is a dashed line indicating $\rho \cdot \tau^2 a = \tau^4 h_1^3$.

6.2. **Some modules over $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$.** We now introduce notation for some modules over $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ which will appear throughout. The reader is encouraged to consult Appendix A to follow along with these definitions.

Definition 6.2. Define the following $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-modules:

- Let $\mathcal{D}$ denote the entire coweight $cw \equiv 1\,(4)$ piece of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$, called a *big diamond* (see Fig. 7);
- Let $\mathcal{S}$ denote the entire coweight $cw \equiv 2\,(4)$ piece of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$, called a *big staircase* (see Fig. 8);
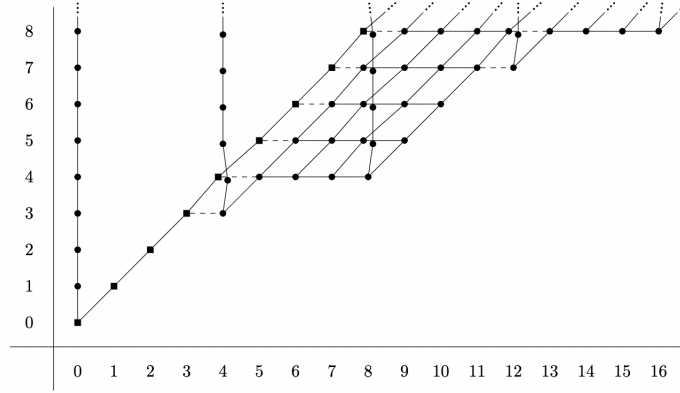
FIGURE 6. $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ in $cw \equiv 0\,(4)$
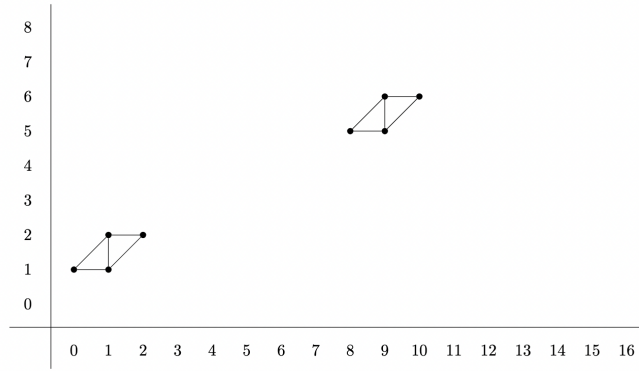
FIGURE 7. $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ in $cw \equiv 1\,(4)$
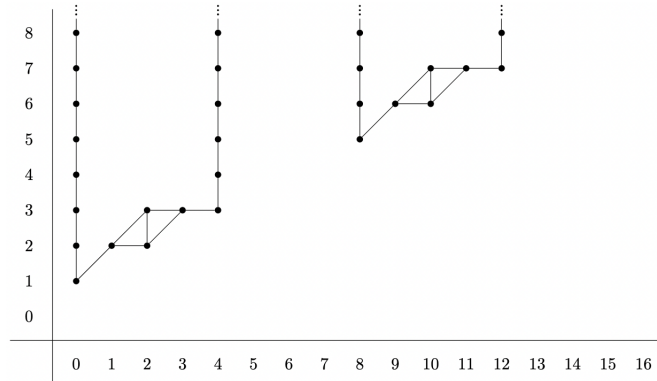
FIGURE 8. $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ in $cw \equiv 2\,(4)$

- Let $P = \mathbb{F}_2[\tau^4][\rho]$, called a ρ-tower, with the obvious $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. 28);

- Let $H = \mathbb{F}_2[\tau^4][h_0]$, called an h_0-*tower*, with the obvious $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. 29);
- Let $PH = \mathbb{F}_2[\tau^4][\rho, h_0]/(\rho h_0)$, called a (ρ, h_0)-*tower*, with the obvious $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. 30);
- Let $D = \mathbb{F}_2[\tau^4][\rho, h_0, h_1]/(\rho^2, h_0^2, h_1^2, \rho h_0, h_0 h_1, \rho h_1 = h_0)$, called a *diamond*, with the obvious $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. 31);
- Let $S = \mathbb{F}_2[\tau^4][h_0, h_1, t, a]/(h_1^3, t^2, a^2, h_0 h_1, h_1 a, ta, h_0^2 t, h_1^2 t)$, called a *staircase*, with the $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module structure given by the relations (see Fig. 32)

$$\rho \cdot t = h_1, \quad h_1 \cdot t = \rho \cdot a, \quad h_0 \cdot t = h_1^2 = \rho^2 \cdot a, \quad \rho^2 \cdot t = 0, \quad \rho^3 \cdot a = 0;$$

- Let $T = \mathbb{F}_2[\tau^4][\rho]/(\rho^2)$, called a *segment*, with the obvious $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. 33);
- Let $J = \mathbb{F}_2[\tau^4][\rho, h_0]/(\rho^3, \rho h_0)$, called a *J-tower*, with the obvious $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. 34);
- Let $JD = \mathbb{F}_2[\tau^4][\rho, h_0, t]/(\rho^3, t^2, \rho^2 t, \rho h_0, h_0^2 t)$, called a *JD-tower*, with the $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module structure given by the relations (see Fig. 35)

$$\rho = h_1 t, \quad h_1 \cdot \rho t = h_0 t = \rho^2.$$

Remark 6.3. As should be indicated by the figures in Appendix A, the naming of these modules is derived from their general shapes when depicted in charts.

We construct now a more intricate family of modules. This module will track the behavior of the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ throughout the process of the coming spectral sequences.

Definition 6.4. In this construction, we will attach generators to extend the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$. We continue until we have attached a generator in Adams filtration 0, at which point we stop. For $n \geq 0$, let $\mathcal{F}_{s,f,w}$ be the $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module concentrated in coweight $cw \equiv s - w\,(4)$, called a *big flag*, described by the following process:

(0) There is a copy of the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ (see Fig. 6) with generator x_0 in degree (s, f, w) called a *flag*;

$(4n+1)$ There is a PH generated by x_{4n+1} in degree $(s - 4 - 8n, f - 1 - 4n, w - 4 - 8n)$ subject to the relations

$$h_1 x_{4n+1} = \rho^3 x_{4n} \text{ and } b x_{4n+1} = \tau^2 a x_{4n}.$$

$(4n+2)$ There is a P generated by x_{4n+2} in degree $(s - 6 + 8n, f - 2 - 4n, w - 6 - 8n)$ subject to to the relations

$$h_1 x_{4n+2} = \rho x_{4n+1} \text{ and } b x_{4n+2} = h_1^2 x_{4n}.$$

$(4n+3)$ There is a P generated by x_{4n+3} in degree $(s - 7 - 8n, f - 3 - 4n, w - 7 - 8n)$ subject to the relations

$$h_1 x_{4n+3} = x_{4n+2} \text{ and } b x_{4n+3} = h_1 x_{4n}.$$

$(4n+4)$ There is a PH generated by x_{4n+4} in degree $(s - 8(n+1), f - 4(n+1), w - 8(n+1))$ subject to the relations

$$h_1 x_{4n+4} = x_{4n+3} \text{ and } b x_{4n+4} = x_{4n}.$$

In short, the submodule $\mathcal{F}_{s,f,w}$ looks like the coweight $cw \equiv 0\,(4)$ portion of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ with generator shifted from $(0, 0, 0)$ to (s, f, w), and then a complicated "tail" pattern coming from the generator of the flag which decreases in all degrees until Adams filtration 0 is reached. Importantly, at each step we are attaching a new generator by an h_1, and since h_1 has coweight 0 as an element of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$, this ensures that each stage of this process is concentrated in the same coweight modulo 4. As an example, we depict the big flag $\mathcal{F}_{12,3,6}$ in Fig. 9.
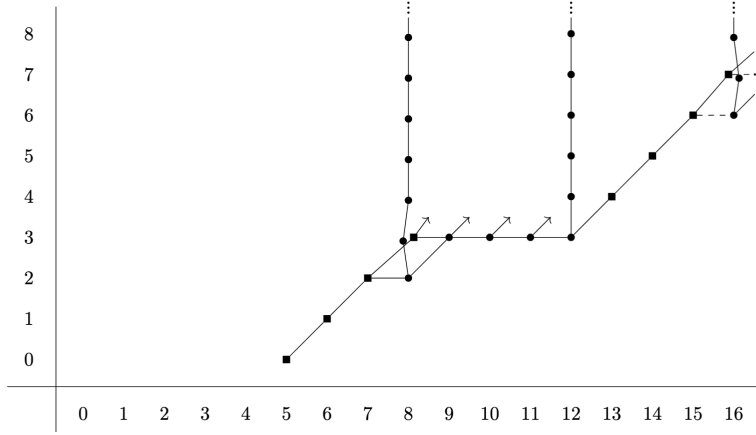
FIGURE 9. The $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module $\mathcal{F}_{12,3,6}$.

Remark 6.5. The big flag $\mathcal{F}_{0,0,0}$ is isomorphic to the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ (see Fig. 6).

6.3. **Comparison with** $\mathbb{C}$. There is a well-studied relationship between $\mathrm{SH}(\mathbb{C})$ and $\mathrm{SH}(\mathbb{R})$ [BI22]. In [Hil11], Hill developed the ρ-Bockstein spectral sequence, an algebraic spectral sequence which allows for the computation of $\mathbb{R}$-motivic information from $\mathbb{C}$-motivic information. The observation that $\mathbb{M}_2^{\mathbb{C}} = \mathbb{M}_2^{\mathbb{R}}/\rho$ lifts to the dual Steenrod algebra, giving $\mathcal{A}_{\mathbb{C}}^{\vee} = \mathcal{A}_{\mathbb{R}}^{\vee}/\rho$. Filtering the cobar complex which computes $\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ by powers of ρ, we obtain the ρ-Bockstein spectral sequence

$$E_1 = \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})[\rho] \implies \mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}).$$

Note that this spectral sequence converges to the E_2-page of the $\mathbb{R}$-motivic **mASS**($\mathbb{S}$). One can also filter the cobar complex which computes $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ by powers of ρ, giving a ρ-Bockstein spectral sequence

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})[\rho] \implies \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}).$$

Note that this spectral sequence converges to the E_2-page of the **mASS**(kq).

There is also an extension of scalars functor $- \otimes \mathbb{C} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(\mathbb{C})$. For $X \in \mathrm{SH}(\mathbb{R})$, we denote its image under extensions of scalars by $X^{\mathbb{C}}$. This functor has the property $\rho \otimes \mathbb{C} = 0$ (modulo 2) and Eilenberg-Mac Lane spectra to are sent to Eilenberg-Mac Lane spectra. In particular, this gives maps

$$\mathbb{M}_2^{\mathbb{R}} \to \mathbb{M}_2^{\mathbb{C}} \cong \mathbb{M}_2^{\mathbb{R}}/\rho, \quad \mathcal{A}_{\mathbb{R}}^{\vee} \to \mathcal{A}_{\mathbb{C}}^{\vee} \cong \mathcal{A}_{\mathbb{R}}^{\vee}/\rho.$$

Extension of scalars induces highly structured maps of Adams spectral sequences, which will allow us to lift $\mathbb{C}$-motivic extensions to $\mathrm{SH}(\mathbb{R})$.

In the 2-complete setting, one may lift ρ to the sphere, giving an element $\rho \in \pi_{-1,-1}^{\mathbb{R}}\mathbb{S}$. This gives a cofiber sequence

$$\mathbb{S}^{-1,-1} \xrightarrow{\rho} \mathbb{S} \to \mathbb{S}/\rho.$$

This cofiber sequence induces a long exact sequence in $\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}$. Since $\rho \otimes \mathbb{C} = 0$, extension of scalars on this long exact sequence gives a split long exact sequence in $\mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}$. Moreover, we can identify $\mathrm{H}_{**}(\mathbb{S}/\rho) \cong \mathbb{M}_2^{\mathbb{R}}/\rho$ since $\mathbb{M}_2^{\mathbb{R}}$ is ρ-torsion free. This gives us the following diagram.

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}) & \xrightarrow{\rho} & \mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}/\rho) & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}) & \xrightarrow{0} & \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}) \oplus \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}) & \longrightarrow & \cdots
\end{array}
$$

The splitting on the bottom row gives a lift $\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}/\rho) \to \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})$. In fact, this map is an isomorphism due to the following.

Proposition 6.6. *Let $X \in \mathrm{SH}(\mathbb{R})$ be any spectrum such that $\mathrm{H}_{**}(X)$ is ρ-torsion free. Then there is an isomorphism*

$$\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathrm{H}_{**}(X/\rho)) \xrightarrow{\cong} \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathrm{H}_{**}(X^{\mathbb{C}})).$$

Proof. First, since $\mathrm{H}_{**}(X)$ is ρ-torsion free, we know that $\mathrm{H}_{**}(X/\rho) \cong \mathrm{H}_{**}(X)/\rho$, where $X/\rho \simeq X \otimes \mathbb{S}/\rho$. This follows from the long exact sequence in homology associated to the cofiber sequence

$$X \xrightarrow{\rho} X \to X/\rho.$$

Note that in this case, we have an isomorphism $\mathrm{H}_{**}(X)/\rho \cong \mathrm{H}_{**}(X^{\mathbb{C}})$. The result now follows from an isomorphism of cobar complexes

$$C_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathrm{H}_{**}(X/\rho)) \cong C_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathrm{H}_{**}(X))/\rho \cong C_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathrm{H}_{**}(X^{\mathbb{C}})).$$

The left isomorphism holds because the cobar complex $C_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathrm{H}_{**}(X/\rho))$ is entirely ρ-torsion free. The right isomorphism holds since $\mathrm{H}_{**}(X)/\rho \cong \mathrm{H}_{**}(X^{\mathbb{C}})$ and $\mathbb{M}_2^{\mathbb{R}}/\rho \cong \mathbb{M}_2^{\mathbb{C}}$. $\square$

One immediate consequence of Proposition 6.6 is that $\mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})$ is a module over $\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathbb{M}_2)$. This module structure is simple to describe. By the ρ-Bockstein, we can represent classes in $\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ in the form $\rho^k x$ for some $x \in \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})$. If $y \in \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})$, then $\rho^k x \cdot y$ is only nonzero when $k = 0$, and in this case we have $x \cdot y = xy$.

Corollary 6.7. *Let $X \in \mathrm{SH}(\mathbb{R})$ be any spectrum such $\mathrm{H}_{**}(X)$ is ρ-torsion free. Then there is an isomorphism*

$$\pi_{**}^{\mathbb{R}}(X/\rho) \cong \pi_{**}^{\mathbb{C}}(X^{\mathbb{C}}).$$

Proof. The map from Proposition 6.6 gives an isomorphism of E_2-pages of Adams spectral sequences. $\square$

Note that $\mathrm{kq}^{\mathbb{C}}$ is the spectrum in $\mathrm{SH}(\mathbb{C})$ representing very effective Hermitian K-theory over $\mathbb{C}$, which we will abusively also denote by kq. Since $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A}_{\mathbb{R}}//\mathcal{A}(1))_{\mathbb{R}}^{\vee}$ is ρ-torsion free, it follows from Corollary 6.7 that $\pi_{**}^{\mathbb{R}}(\mathrm{kq}/\rho) \cong \pi_{**}^{\mathbb{C}}(\mathrm{kq})$. The Künneth spectral sequence for $\mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq})$ collapses by Proposition 2.12, so we also have

$$\pi_{**}^{\mathbb{R}}((\mathrm{kq} \otimes \mathrm{kq})/\rho) \cong \pi_{**}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq}).$$

Combining Corollary 6.7 and Proposition 2.20, we can generalize this idea.

Proposition 6.8. *The ρ-periodic E_1-page of the real kq-resolution is isomorphic to the E_1-page of the complex kq-resolution.*

We can also prove analogous statements using the Brown–Gitler comodules. First, note that we have a diagram in Ext coming from the long exact sequences induced by the cofiber sequence

$$\mathrm{kq} \xrightarrow{\rho} \mathrm{kq} \to \mathrm{kq}/\rho$$

and extension of scalars, combined with the usual change of rings isomorphism:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}) & \xrightarrow{\rho} & \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}/\rho) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}) & \xrightarrow{0} & \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}) \oplus \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}) & \longrightarrow \cdots
\end{array}$$

This gives an isomorphism in Ext by the same argument as before:

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}/\rho) \cong \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}).$$

By construction, the Brown–Gitler comodules $B_0^{\mathbb{R}}(k)$ are ρ-torsion free since they are subcomodules of the dual Steenrod algebra. This extends naturally to tensor factors leading us to the following result:

Proposition 6.9. *There are isomorphisms for all $k \geq 0$:*

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(k)^{\otimes i}/\rho) \cong \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(k)^{\otimes i}).$$

These isomorphisms will be helpful for deducing hidden extensions on the E_∞-page of the **aAHSS**$(B_0^{\mathbb{R}}(k))$ by comparing with the computations of [CQ21] recalled in Section 5.

Remark 6.10. If F is a field with a real embedding, then there is a C_2-equivariant Betti realization functor

$$\mathrm{Be}^{C_2} : \mathrm{SH}(F) \to \mathrm{Sp}^{C_2},$$

obtained from the Betti realization functor of Remark 2.6 by remembering the C_2-action given by complex conjugation. This functor has been well studied, see for example [HO16], and exhibits a close relationship between $\mathbb{R}$-motivic and C_2-equivariant homotopy theory. It was shown by [Kon23] that

$$\mathrm{Be}^{C_2}(\mathrm{kq}_{2,\eta}^{\wedge}) \simeq (\mathrm{ko}_{C_2})_{2,\eta}^{\wedge}.$$

Moreover, there is a decomposition of the homology of a point as $\mathbb{M}_2^{C_2} \cong \mathbb{M}_2^{\mathbb{R}} \oplus NC$, where NC is the so-called "negative cone". This leads to a splitting of the C_2-equivariant dual Steenrod algebra as $\mathcal{A}_{C_2}^{\vee} \cong \mathcal{A}_{\mathbb{R}}^{\vee} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{M}_2^{C_2}$, and hence an isomorphism of Ext groups:

$$\mathrm{Ext}_{\mathcal{A}_{C_2}^{\vee}} \cong \mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}} \oplus \mathrm{Ext}_{NC}.$$

This splitting implies that the E_2-page of the C_2-equivariant Adams spectral sequence computing the ring of cooperation $\pi_\star^{C_2}(\mathrm{ko}_{C_2} \otimes \mathrm{ko}_{C_2})$ contains the E_2-page of the **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$ as a summand. This will be investigated further in joint work with Li, Petersen, and Tatum.

7. The algebraic Atiyah–Hirzebruch spectral sequence

In this section, we compute the group $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(1)^{\otimes i})$ by a series of algebraic Atiyah–Hirzebruch spectral sequences. Motivated by the presentation of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ given in [Gui+20], we organize these computations by coweight modulo 4. Consequently, our results are best displayed in multiple charts which should be read simultaneously. Throughout, we use the notation of Section 6.

7.1. $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(1))$. Now, we will compute $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(1))$ by the **aAHSS**$(B_0^{\mathbb{R}}(1))$. This spectral sequence takes the form

$$E_1^{s,f,w,a} = \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}^{s,f,w}(\mathbb{M}_2^{\mathbb{R}}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{M}_2^{\mathbb{R}}\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\} \implies \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}^{s,f,w}(B_0^{\mathbb{R}}(1)).$$

The d_1 and d_2 differentials in this spectral sequence are determined by our discussion in Section 4.2. In particular the d_1-differential is only nonzero on Atiyah–Hirzebruch filtration 3, where

$$d_1(\alpha[3]) = h_0\alpha[2],$$

and the d_2-differential is only nonzero on Atiyah–Hirzebruch filtration 3, where

$$d_2(\alpha[2]) = h_1\alpha[0].$$

Note that in terms of coweight, both of these differentials are determined by multiplication by an element in $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ of coweight 0. Given the complicated nature of this spectral sequence, we only present charts where the differential is nonzero and only include the relevant Atiyah–Hirzebruch filtrations. For example, since the d_1-differential is trivial on Atiyah–Hirzebruch filtration 0, we omit this filtration from the charts for the E_1-page.

Fig. 10 depicts the E_1-page with coweight $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtrations 2 and 3. Fig. 11 depicts the E_1-page with coweight $cw \equiv 1\,(4)$ in Atyiah-Hirzebruch filtrations 2 and 3. Fig. 12 depicts the E_1-page with coweight $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtrations 2 and 3. Fig. 13 depicts the E_2-page with coweight $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtrations 0 and 2. Fig. 14 depicts the

E_2-page with coweight $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtrations 0 and 2. Fig. 15 depicts the E_2-page with coweight $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtrations 0 and 2. Differentials are blue, linear with respect to the underlying $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module structure, and preserve motivic weight.
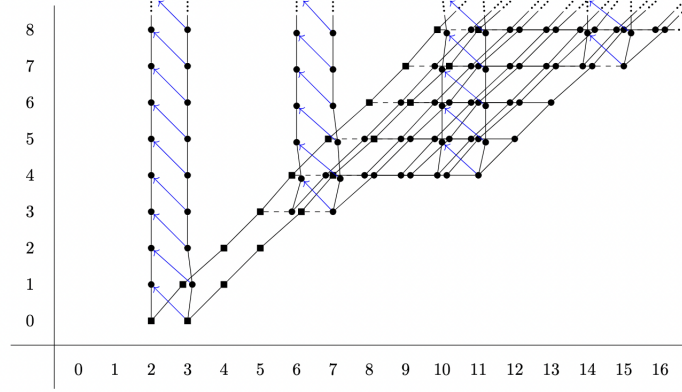

FIGURE 10. The E_1-page of the **aAHSS**$(B_0^{\mathbb{R}}(1))$ with $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtration 2 and 3.
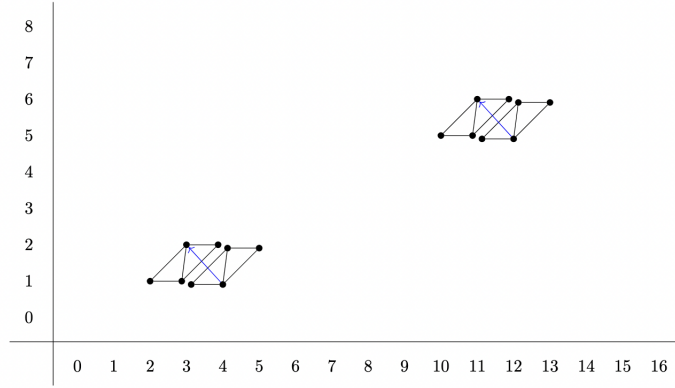
FIGURE 11. The E_1-page of the **aAHSS**$(B_0^{\mathbb{R}}(1))$ with $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 2 and 3.

As we remarked in Section 4.2, the d_3-differential in this spectral sequence is dependent on the field we are working over. While both classically and $\mathbb{C}$-motivically we saw that the d_3-differential, which is only potentially nonzero on Atiyah–Hirzebruch filtration 3, was actually trivial for degree reasons, we see that this is not the case here. By inspection, the only possible nonzero d_3 differential is between Atiyah–Hirzebruch filtrations 3 and 0, and will be given by the formula

$$d_3(\alpha[3]) = \langle \alpha, h_0, h_1 \rangle[0].$$

Lemma 7.1. *There is a d_3-differential in the* **aAHSS**$(B_0^{\mathbb{R}}(1))$*:*

$$d_3(\rho[3]) = (\tau h_1)[0].$$

Proof. This differential is witnessed by the Massey product $\langle \rho, h_0, h_1 \rangle$. This Massey product is shown to be equal to τh_1 in [Gui+20, Thm 8.1] with zero indeterminacy. $\qquad\square$
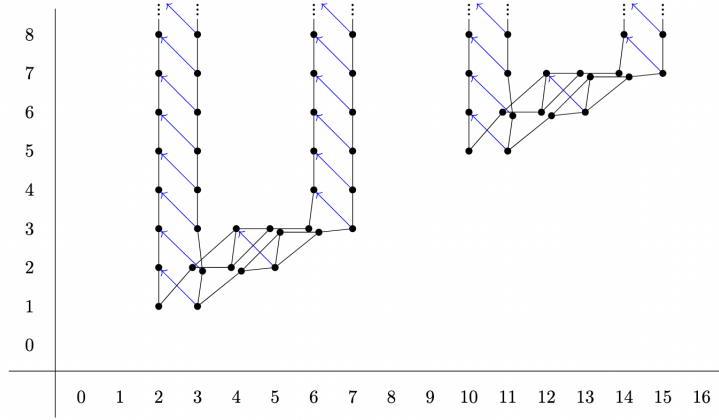
FIGURE 12. The E_1-page of the **aAHSS**$(B_0^{\mathbb{R}}(1))$ with $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtration 2 and 3.
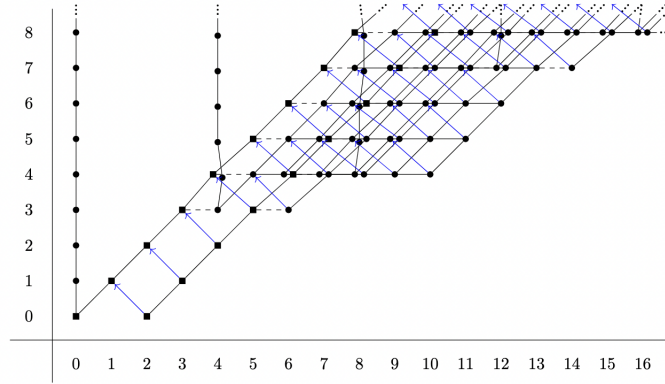

FIGURE 13. The E_2-page of the **aAHSS**$(B_0^{\mathbb{R}}(1))$ with $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtration 0 and 2.

In fact, since the differentials are linear over $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ this determines all d_3-differentials. For example, we have

$$d_3(b\rho[3]) = b \cdot \tau h_1[0],$$

and we have

$$d_3(\rho \cdot \tau h_1[3]) = (\tau h_1)^2[0].$$

It is important to note that since τh_1 has coweight 1 in $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$, and since each Atiyah–Hirzebruch filtration is zero in coweight 3, the d_3-differential is trivial on the coweight $cw \equiv 2\,(4)$ portion of Atiyah–Hirzebruch filtration 3. This coweight jump requires us to pay close attention to the d_3-differential. For example, in the coweight $cw \equiv 0\,(4)$ piece of Atiyah–Hirzebruch filtration 3, there is an infinite ρ-tower stemming from $\rho[3]$. However, while

$$d_3(\rho[3]) = \tau h_1[0]$$
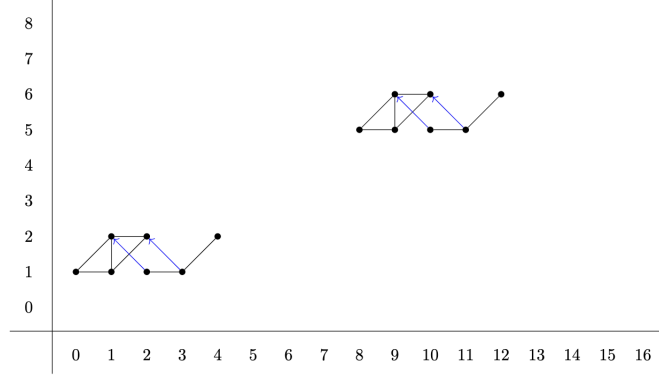
and

$$d_3(\rho^2[3]) = \rho \cdot \tau h_1[0],$$

FIGURE 14. The E_2-page of the **aAHSS**($B_0^{\mathbb{R}}(1)$) with $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 0 and 2.
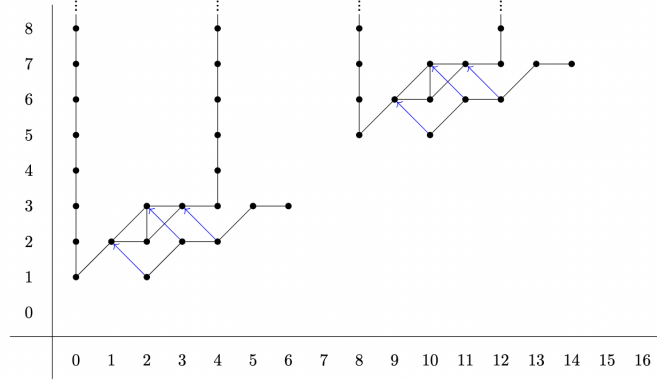

FIGURE 15. The E_2-page of the **aAHSS**($B_0^{\mathbb{R}}(1)$) with $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtration 0 and 2.

there are no other classes in the coweight $cw \equiv 1\,(4)$ piece of Atiyah–Hirzebruch filtration 0 for the rest of this ρ-tower to hit. In other words, for we have

$$d_3(\rho^n[3]) = 0[0],\ n \geq 2.$$

Fig. 16 depicts the E_3-page where the coweight of Atiyah–Hirzebruch filtration 3 is $cw \equiv 0\,(4)$ and the coweight of Atiyah–Hirzebruch filtration 0 is $cw \equiv 1\,(4)$. Fig. 17 depicts the E_3-page where the coweight of Atiyah–Hirzebruch filtration 3 is $cw \equiv 1\,(4)$ and the coweight of Atiyah–Hirzebruch filtration 0 is $cw \equiv 2\,(4)$.

Recall from Proposition 4.6 that $E_4 = E_\infty$ for Atiyah–Hirzebruch filtration reasons. Hidden extensions can be realized using complex Betti realization, linearity over $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$, base change $\otimes \mathbb{C} : \mathrm{SH}(\mathbb{C}) \to \mathrm{SH}(\mathbb{R})$, and the isomorphism from Proposition 6.9:

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(1)/\rho) \cong \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(1)).$$

The following result will be useful when depicting the E_∞-page in charts.

Lemma 7.2. *The group* $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(1))$ *is zero in coweight* $cw \equiv 1\,(4)$.

Proof. In coweight $cw \equiv 1\,(4)$, the E_∞-page of the **aAHSS**($B_0^{\mathbb{R}}(1)$) is isomorphic to the a direct sum of coweight $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 0, coweight $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch
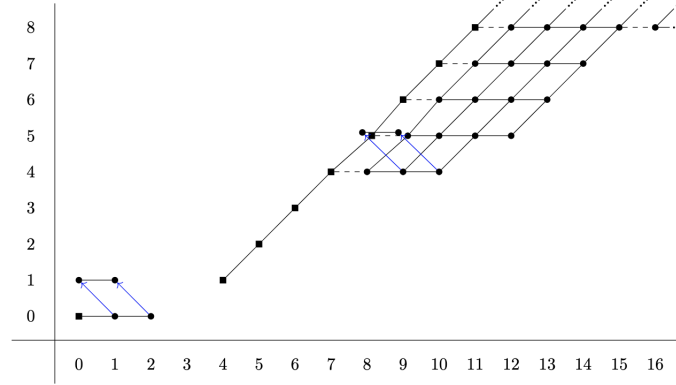
FIGURE 16. The E_3-page of the **aAHSS**$(B_0^{\mathbb{R}}(1))$ with $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtration 3 and $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 0.
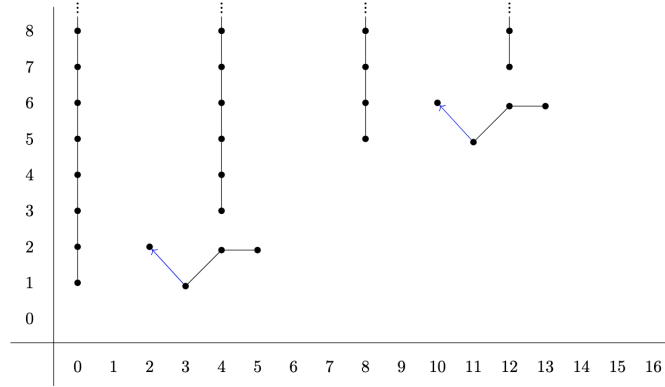

FIGURE 17. The E_3-page of the **aAHSS**$(B_0^{\mathbb{R}}(1))$ with $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 3 and $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtration 0.

filtration 2, and coweight $cw \equiv 3\,(4)$ in Atiyah–Hirzebruch filtration 3 pieces. The differentials:

$$d_2(b^n \cdot \tau h_1[2]) = b^n h_1 \cdot \tau h_1[0], \quad d_2(b^n \rho \cdot \tau h_1[2]) = b^n \rho h_1 \cdot \tau h_1[0], \quad n \geq 0$$

and

$$d_3(b^n \rho[3]) = b^n \cdot \tau h_1[0], \quad d_3(b^n \rho^2[3]) = b^n \rho \cdot \tau h_1[0], \quad n \geq 0$$

ensure that Atiyah–Hirzebruch filtration 0 is trivial in coweight $cw \equiv 1\,(4)$. The differentials

$$d_1((\tau^2 a)^n b^m[3]) = (\tau^2 a)^n b^m h_0[2], \quad d_2((\tau^2 a)^n b^m[2]) = (\tau^2 a)^n b^m h_1[0], \quad n, m \geq 0$$

ensure that Atiyah–Hirzebruch filtration 2 is trivial in coweight $cw \equiv 0\,(4)$. Atiyah–Hirzebruch filtration 3 is trivial in coweight $cw \equiv 3\,(4)$ since $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(\mathbb{M}_2^{\mathbb{R}})$ is trivial in this coweight. $\qquad\square$

We organize the E_∞-page by coweight modulo 4 in the same way that we depicted $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(\mathbb{M}_2^{\mathbb{R}})$. As indicated by Lemma 7.2, the different Atiyah–Hirzebruch filtration pieces contribute different pieces to each coweight of the E_∞-page, suitably shifted by the degree of the cell in the filtration it is attached to. Since $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(\mathbb{M}_2^{\mathbb{R}})$ is concentrated in coweights $cw \equiv 0, 1$, and $2\,(4)$, for each Atiyah–Hirzebruch filtration piece there is a coweight on the E_∞-page which it does not contribute to. To make this clear, we describe each coweight piece of the E_∞-page in terms of contributions from Atiyah–Hirzebruch filtration in Table 3.

E_∞ coweight	Filtration 0	Filtration 2	Filtration 3
0	0	3	2
1	1	0	3
2	2	1	0
3	3	2	1

TABLE 3. Atiyah–Hirzebruch filtration coweight contributions in the **aAHSS**($B_0^{\mathbb{R}}(1)$).

With this in mind, we are prepared to give charts for the E_∞-page. Fig. 18 depicts the E_∞-page in coweight $cw \equiv 0\,(4)$. Fig. 19 depicts the E_∞-page in coweight $cw \equiv 2\,(4)$. Fig. 20 depicts the E_∞-page in coweight $cw \equiv 3\,(4)$. We use the same notation as previously used in this section, with some additions. A $\square$ represents $\mathbb{F}_2[\tau^4][\rho]$, where in contrast to a black $\blacksquare$, the ρ-tower does not support h_1-multiplication. Red lines indicate hidden extensions. A diagonal black $\nearrow$ indicates infinite h_1-multiplication on a ρ-tower which is compatible with an infinite ρ-tower off of a class which is h_1-divisible. For example, in coweight $cw \equiv 2\,(4)$, there is a class x in degree $(3,1)$ which supports an infinite h_1-tower, and there is a class y in degree $(5,2)$ which is part of an infinite h_1-tower and supports an infinite ρ-tower. Hidden in our charts, and as indicated by $\nearrow$, is the relation $h_1 x = \rho y$.
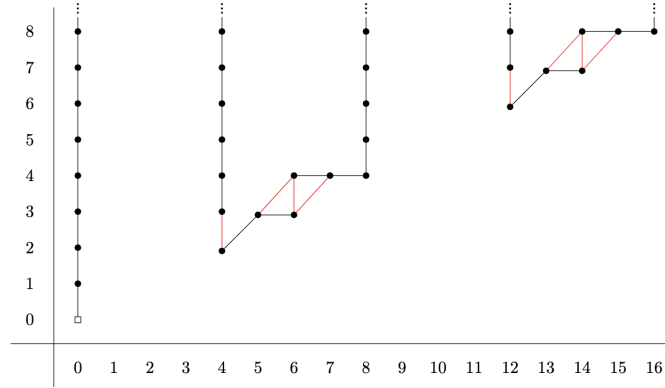
FIGURE 18. The E_∞-page of the **aAHSS**($B_0^{\mathbb{R}}(1)$) in $cw \equiv 0\,(4)$.
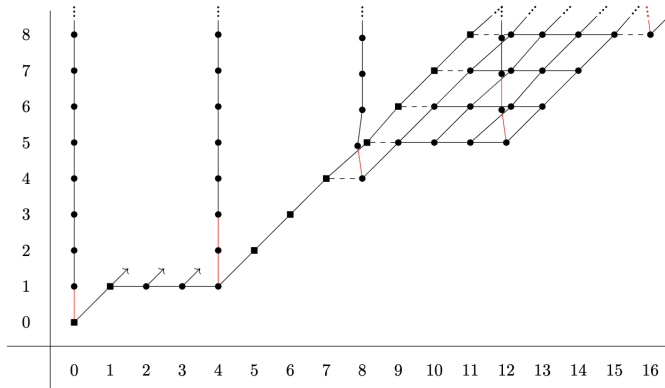
FIGURE 19. The E_∞-page of the **aAHSS**($B_0^{\mathbb{R}}(1)$) in $cw \equiv 2\,(4)$.
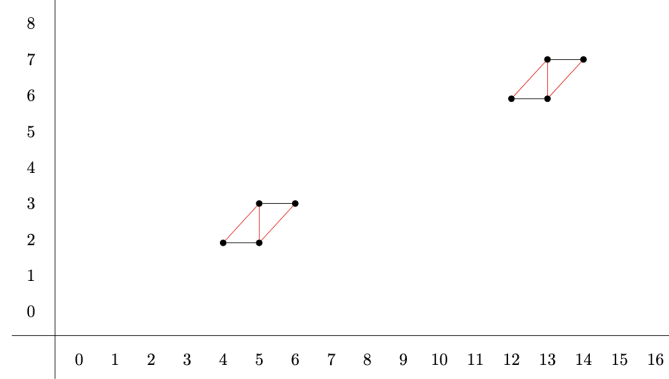
FIGURE 20. The E_∞-page of the **aAHSS**$(B_0^{\mathbb{R}}(1))$ in $cw \equiv 3\,(4)$.

Remark 7.3. The big flag $\mathcal{F}_{4,1,2}$ is isomorphic to the coweight $cw \equiv 2\,(4)$ piece of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1))$ (see Fig. 19).

We note that most classes on the E_∞-page are v_1-periodic. In fact, the only classes which are v_1-torsion are those in the ρ-tower in Adams filtration 0 of the coweight $cw \equiv 0\,(4)$ piece which are divisible by ρ^3.

We close this section with the following result.

Corollary 7.4. *There is an isomorphism*

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1)) \cong \pi_{**}^{\mathbb{R}}(\mathrm{ksp}).$$

Proof. The proof is identical to the one given in Corollary 5.4. $\qquad\square$

7.2. $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1)^{\otimes 2})$ and recursive submodules.

To illustrate the numerous v_1-torsion classes which arise in the process of computing $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$, which are ultimately to be ignored (see Remark 4.3), we explicitly compute $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$. We form an algebraic Atiyah–Hirzebruch spectral sequence by taking the cellular filtration on $B_0^{\mathbb{R}}(1)$ from before and applying the functor $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}\left(B_0^{\mathbb{R}}(1) \otimes -\right)$. This has signature

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1)) \otimes \mathbb{M}_2^{\mathbb{R}}\{1, \bar{\xi}_1, \bar{\tau}_1\} \implies \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1)^{\otimes 2}).$$

The differentials are determined in an analogous manner as before, and hidden extensions can be realized by the same methods. Similar to the previous case, the behavior of the differentials ensures that the E_∞-page is trivial in coweight $cw \equiv 1\,(4)$. We present the E_∞ page, modulo v_1-torsion, below. Fig. 21 depicts the E_∞-page in coweight $cw \equiv 0\,(4)$. Fig. 22 depicts the E_∞-page in coweight $cw \equiv 1\,(4)$. Fig. 23 depicts the E_∞-page in coweight $cw \equiv 2\,(4)$. Note that we do not present hidden extensions in red in these charts.

Remark 7.5. The big flag $\mathcal{F}_{8,2,4}$ is isomorphic to the summand of the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1)^{\otimes 2})$ concentrated in stems $s \geq 2$ (see Fig. 21).

Before giving a general formula for $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$, we make a few simple observations. The group $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(\mathbb{M}_2^{\mathbb{R}})$, is only nonzero in coweights $cw \equiv 0, 1, 2\,(4)$. After running the **aAHSS**$(B_0^{\mathbb{R}}(1))$, we see that there is a submodule

$$M_1 := \Sigma^{4,2}\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(\mathbb{M}_2^{\mathbb{R}})\langle 1 \rangle \subseteq \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1)),$$

using the notation from Section 5. Note that if x has coweight $cw \equiv t\,(4)$, then $\Sigma^{4,2}x\langle 1 \rangle$ has coweight $cw \equiv t + 2\,(4)$. In particular, M_1 is concentrated in coweights $cw \equiv 0, 2, 3\,(4)$. By observation, we
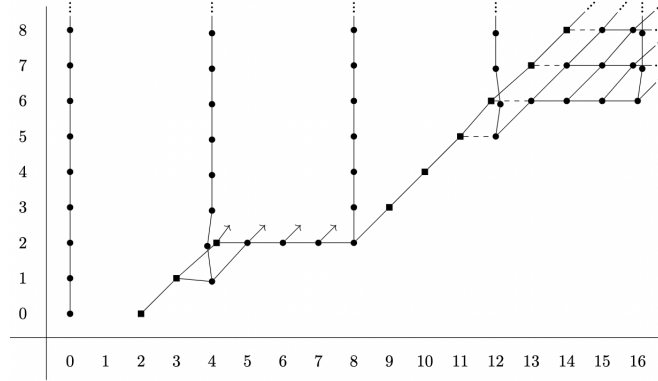
FIGURE 21. The E_∞-page of the **aAHSS**$\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ in $cw \equiv 0\,(4)$, modulo v_1-torsion.
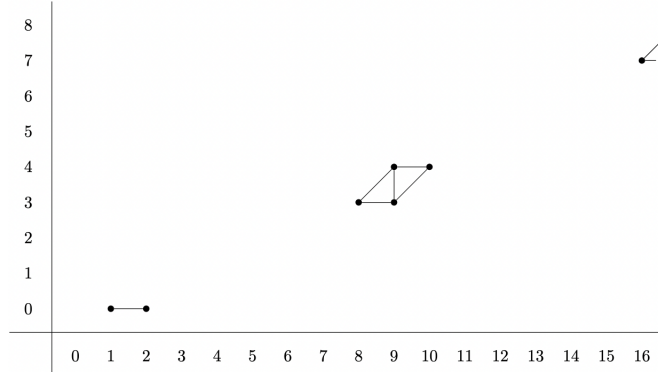

FIGURE 22. The E_∞-page of the **aAHSS**$\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ in $cw \equiv 1\,(4)$, modulo v_1-torsion.
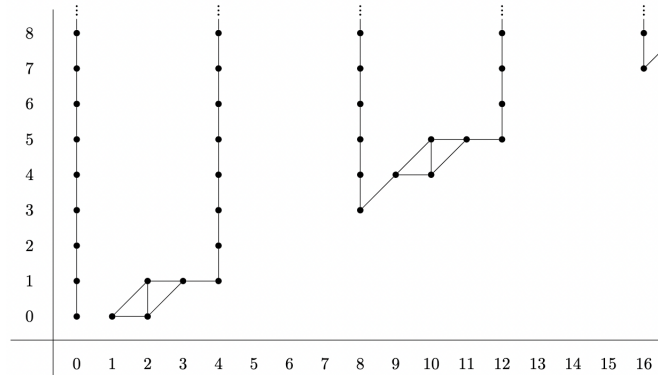

FIGURE 23. The E_∞-page of the **aAHSS**$\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ in $cw \equiv 2\,(4)$, modulo v_1-torsion.

see that in fact all classes in $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1))$ are concentrated in these coweights. Similarly, after running the **aAHSS**$\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ we see that there is a submodule

$$M_2 := \Sigma^{8,4}\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(\mathbb{M}_2^{\mathbb{R}})\langle 2\rangle \subseteq \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right).$$

It follows that M_2 is concentrated in coweights $cw \equiv 0, 1, 2\,(4)$, and by observation we see that in fact all classes in $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ are concentrated in these coweights. More generally, from the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$ we see that the group $\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(1)^{\otimes i})$ contains a submodule

$$M_i = \Sigma^{4i,2i}\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})\langle i \rangle \subseteq \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right),$$

and that, modulo v_1-torsion, $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$ is concentrated in the same coweights as M_n is. More precisely, we have the following result.

Proposition 7.6. *The group* $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)/v_1\text{-torsion is trivial in coweight } cw \equiv 1\,(4) \text{ for}$ $n \equiv 1\,(2)$ *and in coweight* $cw \equiv 3\,(4)$ *for* $n \equiv 0\,(2)$.

We also have the following submodule of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(1)^{\otimes i})$.

Proposition 7.7. *Let* $i \geq 1$. *For all* $1 \leq j \leq i$, *the group* $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)/v_1\text{-torsion contains}$ *as a submodule:*

$$\Sigma^{4i,2i}\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes j}\right)\langle i \rangle \subseteq \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right).$$

Proof. We have shown this to be true in the case of $i = 1$ in Corollary 7.4. The differentials in the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ on the submodule M_1 are isomorphic to the differentials in the **aAHSS** $(B_0^{\mathbb{R}}(1))$, hence the submodule M_2 of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ extends to give a submodule

$$\Sigma^{8,4}\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)\right)\langle 2 \rangle \subseteq \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right).$$

For arbitrary i, the submodule $M_{i-1} \subseteq \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes i-1}\right)$ extends to the submodule

$$\Sigma^{4(i-1),2(i-1)}\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes j}\right)\langle i-1 \rangle$$

for each $1 \leq j \leq i-1$. The differentials in the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$ are isomorphic to the differentials in the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes j}\right)$ when restricted to this submodule, hence the result of the spectral sequence gives the desired submodules in the target. $\qquad\square$

Remark 7.8. It is important that we work modulo v_1-torsion to make the above claim. For example, by computing the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes 3}\right)$ one can observe that the group $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes 3}\right)$ is nonzero in every coweight modulo 4. However, the only classes in coweight $cw \equiv 1\,(4)$ are v_1-torsion for degree reasons. Moreover, these v_1-torsion classes are permanent cycles of the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes 4}\right)$, so their impact on this inductive process is negligible

7.3. **Ext**$_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$. To aid in our presentation of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$, we use the notation from Section 6.1 to construct a family of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-modules which we organize by coweight modulo 4. As we will see, each module is concentrated in precisely 3 of these coweights, reflecting our observations from Proposition 7.6.

Definition 7.9. For each $i \geq 0$, let $Z_i^{\mathbb{R}}$ be the $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module given by the direct sum of the columns of the corresponding table.

For $i = 4k$, let $Z_i^{\mathbb{R}}$ be the direct sum of the columns of Table 4:

$cw \equiv 0\,(4)$	$cw \equiv 1\,(4)$	$cw \equiv 2\,(4)$
$\mathcal{F}_{4i,i,2i}$	$\Sigma^{2i,i}\mathcal{D}$	$\Sigma^{2i,i}\mathbb{S}$
$\oplus$		$\oplus$
$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor (j+1)/2 \rfloor -4}H$		$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2 \rfloor -2}H$

TABLE 4. $Z_i^{\mathbb{R}}$ for $i = 4k$

For $i = 4k+1$, let $Z_i^{\mathbb{R}}$ be the direct sum of the columns of Table 5:

$cw \equiv 0\,(4)$	$cw \equiv 2\,(4)$	$cw \equiv 3\,(4)$
$\Sigma^{2i+2,i+1}\mathcal{S}\langle 1\rangle$	$\mathcal{F}_{4i,i,2i}$	$\Sigma^{2i+2,i+1}\mathcal{D}\langle 1\rangle$
$\oplus$	$\oplus$	
$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor (j+1)/2\rfloor -4}H$	$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2\rfloor -2}H$	
$\oplus$		
$\Sigma^{2i+2,i-1}J$		

TABLE 5. $Z_i^{\mathbb{R}}$ for $i = 4k+1$

For $i = 4k+2$, let $Z_i^{\mathbb{R}}$ be the direct sum of the columns of Table 6:

$cw \equiv 0\,(4)$	$cw \equiv 1\,(4)$	$cw \equiv 2\,(4)$
$\mathcal{F}_{4i,i,2i}$	$\Sigma^{2i+4,i+2}\mathcal{D}\langle 2\rangle$	$\Sigma^{2i+4,i+2}\mathcal{S}\langle 2\rangle$
$\oplus$	$\oplus$	$\oplus$
$\bigoplus_{j=1}^{2k+1} \Sigma^{4j-4,4\lfloor (j+1)/2\rfloor -4}H$	$\Sigma^{2i-2,i-1}T$	$\bigoplus_{j=1}^{2k+1} \Sigma^{4j-4,4\lfloor j/2\rfloor -2}H$
		$\oplus$
		$\Sigma^{2i,i}JD\langle 1\rangle$

TABLE 6. $Z_i^{\mathbb{R}}$ for $i = 4k+2$

For $i = 4k+3$, let $Z_i^{\mathbb{R}}$ be the direct sum of the columns of Table 7:

$cw \equiv 0\,(4)$	$cw \equiv 2\,(4)$	$cw \equiv 3\,(4)$
$\Sigma^{2i-2,i-1}\mathcal{S}\langle -1\rangle$	$\mathcal{F}_{4i,i,2i}$	$\Sigma^{2i-2,i-1}\mathcal{D}\langle -1\rangle$
$\oplus$	$\oplus$	
$\bigoplus_{j=1}^{2k+1} \Sigma^{4j-4,4\lfloor j/2\rfloor}H$	$\bigoplus_{j=1}^{2k+2} \Sigma^{4j-4,4\lfloor (j+1)/2\rfloor -2}H$	

TABLE 7. $Z_i^{\mathbb{R}}$ for $i = 4k+3$

We are now prepared to state the $\mathbb{R}$-motivic analogue of Proposition 5.5.

Theorem 7.10. *There is an isomorphism of $\mathcal{A}(1)_{\mathbb{R}}^\vee$-comodules and $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(\mathbb{M}_2^{\mathbb{R}})$-modules:*

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)}{v_1\text{-torsion}} \cong Z_i^{\mathbb{R}}$$

Proof. The case of $i = 1$ was computed in Corollary 7.4, and the case of $i = 2$ is presented in Fig. 21, Fig. 22, and Fig. 23. The general case is obtained by applying the functor $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1)^{\otimes i-1})$ to the cellular filtration of $B_0^{\mathbb{R}}(1)$ and computing the **aAHSS**$(B_0^{\mathbb{R}}(1)^{\otimes i})$:

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1)^{\otimes i-1}) \otimes \mathbb{M}_2^{\mathbb{R}}\{[1], [\bar{\xi}_1], [\bar{\tau}_1]\} \implies \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(1)^{\otimes i}).$$

The differentials are determined by the attaching maps in the algebraic cell complex $B_0^{\mathbb{R}}(1)$ as before, hence Proposition 4.6 implies that $E_4 = E_\infty$. Hidden extensions may be obtained using complex Betti realization to the classical case, linearity over $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(\mathbb{M}_2^{\mathbb{R}})$, and the base change functor $\otimes\mathbb{C} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(\mathbb{C})$ combined with the isomorphism from Proposition 6.9:

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^\vee}(B_0^{\mathbb{R}}(k)^{\otimes i}/\rho) \cong \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{C}}^\vee}(B_0^{\mathbb{C}}(k)^{\otimes i}).$$

The calculations are very similar for all i. We give the full argument in the case of $4k+1$ and leave the rest to the reader. As an example, we depict $Z_5^{\mathbb{R}}$ in Fig. 24, Fig. 25, and Fig. 26.

Let $i = 4k$, and consider the **aAHSS**$(B_0^{\mathbb{R}}(1)^{\otimes 4k+1})$. By induction, the E_1-page is isomorphic to

$$Z_{4k}^{\mathbb{R}} \otimes \mathbb{M}_2^{\mathbb{R}}\{[1], [\bar{\xi}_1], [\bar{\tau}_1]\}.$$

Recall from Section 7.1 the contributions from each Atiyah–Hirzebruch filtration to each coweight of the E_∞-page of the spectral sequence. In particular, since $Z_{4k}^{\mathbb{R}}$ is trivial in coweight $cw \equiv 3\,(4)$, the E_∞-page consist of summands from only two Atiyah–Hirzebruch filtration pieces in coweights $cw \equiv 0, 1, 3\,(4)$. We proceed by analyzing the E_∞-page of the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes 4k+1}\right)$ one coweight at a time.

Coweight $cw \equiv 0\,(4)$

We begin by analyzing the coweight $cw \equiv 0\,(4)$ portion of the spectral sequence. This consists of classes coming from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0\,(4)$ and from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 2\,(4)$. There is a d_1-differential between the h_0-towers of Atiyah–Hirzebruch filtrations 2 and 3 in coweight $cw \equiv 2\,(4)$. Letting x_j be a generator for $\Sigma^{4j-4,4\lfloor j/2 \rfloor - 2}H$, this differential is determined by

$$d_1(x_j[3]) = h_0 \cdot x_j[2].$$

There is also a d_1-differential between big staircases $\Sigma^{2i,i}\mathbb{S}$ in Atiyah–Hirzebruch filtrations 2 and 3 in coweight $cw \equiv 2\,(4)$. The classes in each staircase in Atiyah–Hirzebruch filtration 3 which do not support a d_1 differential are those which do not support an h_0-tower. These classes cannot support any further differentials for degree reasons, and so they survive to the E_∞-page.

The d_2-differential between Atiyah–Hirzebruch filtrations 0 and 2 is given by the h_1-attaching map in $B_0^{\mathbb{R}}(1)$. In particular, since the h_0-towers from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0\,(4)$ do not support h_1 multiplication, each tower survives to E_3. None of these h_0-towers can be the target of a d_3 differentials for degree reasons, so they also survive to the E_∞-page. The d_2-differential between the big flags $\mathcal{F}_{4i,i,2i}$ in Atiyah–Hirzebruch filtration 2 and 3 in coweight $cw \equiv 0\,(4)$ is determined by the structure given in Definition 6.4. In particular, the classes in the big flag in Atiyah–Hirzebruch filtration 0 which survive to E_3 are those classes which are not h_1-divisible. For $i = 4k$, these are precisely the h_0-towers in stems $s \equiv i + 4\,(8)$, the J-towers in stems $s \equiv i\,(8)$ with Adams filtration $f \geq 1$, and the (ρ, h_0)-tower in Adams filtration $f = 0$. For degree reasons, the d_3-differential is trivial on Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0\,(4)$.

Thus, in coweight $cw \equiv 0$, the E_∞-page is given by the h_0-towers from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0,\,(4)$ which do not come from the big flag:

$$\bigoplus_{k=0}^{2k} \Sigma^{4j-4,4\lfloor (j+1)/2 \rfloor - 4}H;$$

the h_0-towers, J-towers and (ρ, h_0)-tower remaining from the big flag $\mathcal{F}_{4i,i,2i}$ in Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0\,(4)$; and the remaining classes from the big staircase $\Sigma^{2i,i}\mathbb{S}$ in Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 2\,(4)$. Base change to $\mathbb{C}$ and linearity give hidden extensions between the h_0-towers and J-towers from Atiyah–Hirzebruch filtration 0 and the remaining classes from the big staircase in Atiyah–Hirzebruch filtration 3. To be precise, the classes in $\Sigma^{2i,i}\mathbb{S}[3]$ in smallest stem and Adams filtration which survive to E_∞ are represented by

$$\Sigma^{2i,i}(h_1 \cdot (\tau^2 h_0))[3], \quad \Sigma^{2i,i}(h_1^2 \cdot (\tau^2 h_0))[3], \quad \Sigma^{2i,i}(\rho a)[3],$$

and their τ^4-multiples. Let $x_{i-4}[0]$ denote the generator of $H \subseteq \mathcal{F}_{4i,i,2i}$ from Atiyah–Hirzebruch filtration 0 in lowest stem and Adams-filtration, and let $x_{i-5}[0]$ denote the generator of $J \subseteq \mathcal{F}_{4i,i,2i}$ from the same Atiyah–Hizebruch filtration. Base change to $\mathbb{C}$ gives hidden extensions

$$h_0 \cdot \Sigma^{2i,i}(h_1 \cdot (\tau^2 h_0))[3] = x_{i-4}[0], \quad h_1 \cdot \Sigma^{2i,i}(\rho a)[3] = \rho x_{i-5}[0].$$

Then, linearity over $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$ gives us that

$$h_1 \cdot \Sigma^{2i,i}(h_1^2 \cdot \tau^2 h_0)[3] = h_1 \cdot \Sigma^{2i,i}(\rho^2 a)[3] = \rho \cdot h_1 \cdot \Sigma^{2i,i}(\rho a)[3] = \rho^2 x_{i-5}[0]$$

and

$$h_0 \cdot \Sigma^{2i,i}(\rho a)[3] = h_0 \cdot \Sigma^{2i,i}(h_1(\tau h_1)^2)[3] = h_1 \cdot \Sigma^{2i,i}(h_0(\tau h_1)^2)[3] = h_1 \cdot \Sigma^{2i,i}(\rho^2 a)[3] = \rho^2 x_{i-5}[0].$$

Multiplication by b gives the same extensions throughout the E_{∞}-page. Altogether, this gives the summand

$$\Sigma^{2i+4,i+2}\mathbb{S}\langle 1 \rangle.$$

For degree reasons, the elements in the (ρ, h_0)-tower from Atiyah–Hirzebruch filtration 0 which are divisible by ρ^3 must be v_1-torsion by inspection. Since our result is modulo this torsion, this gives the last summand
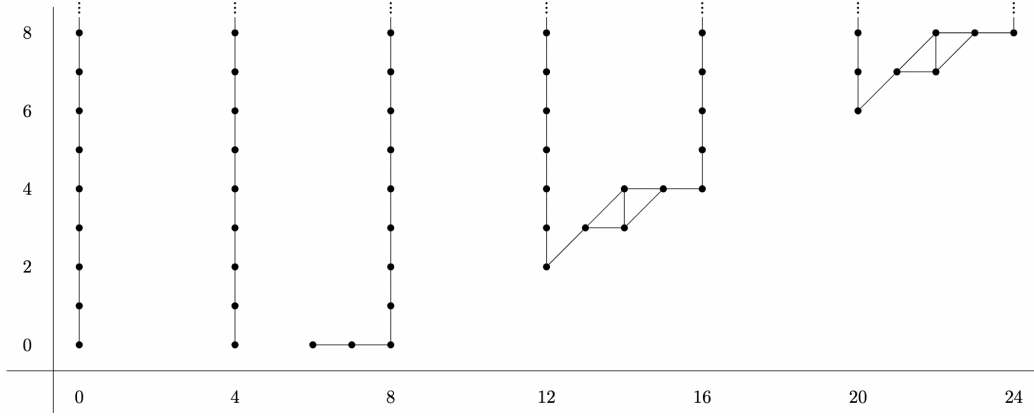
$$\Sigma^{2i-2,i-1}J.$$

FIGURE 24. $Z_5^{\mathbb{R}}$ in $cw \equiv 0 \, (4)$.

Coweight $cw \equiv 1 \, (4)$

The coweight $cw \equiv 1 \, (4)$ portion of the spectral sequence consists of classes coming from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 1 \, (4)$ and from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 0 \, (4)$. The d_1-differential between Atiyah–Hirzebruch filtrations 2 and 3 kills all of the classes from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 0 \, (4)$ which are divisible by h_0. The d_2-differential between Atiyah–Hirzebruch filtrations 0 and 2 kills all of the classes from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 0 \, (4)$ which support multiplication by h_1. By observation, we see that every class remaining from the big flag $\mathcal{F}_{4i,i,2i}$ on the E_2-page is of this kind. Moreover, the filtration forces $E_3 = E_{\infty}$ on Atiyah–Hirzebruch filtration 2, so the only classes which survive are the bases of the h_0-towers

$$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor (j+1)/2 \rfloor - 4}\mathbb{F}_2[\tau^4].$$

The d_2-differential between Atiyah–Hirzebruch filtrations 0 and 2 kills all of the classes from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 1 \, (4)$ which are divisible by h_1. This can be recovered by appropriately shifting the analogous d_2-differential in the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ (see Fig. 14). The d_3-differential between Atiyah–Hirzebruch filtrations 0 and 3 is determined by the formula $d_3(\rho[3]) = (\tau h_1)[0]$ established in Lemma 7.1. In particular, each remaining class from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 1$ is the target of a differential. Similar to the d_2-differential just mentioned, this can be recovered by appropriately shifting the analogous d_3-differential in

the **aAHSS**$(B_0^{\mathbb{R}}(1))$ (see Fig. 16). Thus, there are no contributions to the E_∞-page in coweight $cw \equiv 1\,(4)$ from Atiyah–Hirzebruch filtration 0. Notice that this forces the classes in the summand $\bigoplus_{j=0}^{2k} \Sigma^{4j-4,4\lfloor(j+1)/2\rfloor-4}\mathbb{F}_2[\tau^4]$ to be v_1-torsion, hence the E_∞-page is trivial in coweight $cw \equiv 1\,(4)$.

Coweight $cw \equiv 2\,(4)$

The coweight $cw \equiv 2\,(4)$ portion of the spectral sequence consists of classes coming from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$, from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 1\,(4)$, and from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 0\,(4)$. The d_2-differential between Atiyah–Hirzebruch filtrations 0 and 2 kills all of the classes from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$ which are divisible by h_1. Notice that while the behavior on the big staircase $\Sigma^{2i,i}\mathbb{S}$ can be recovered from the analogous differential in the **aAHSS**$(B_0^{\mathbb{R}}(1))$ (see Fig. 15), the summand

$$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2\rfloor-2}H$$

in Atiyah–Hirzebruch filtration 0 is entirely h_1-torsion, hence survives to E_3. The d_3-differential between Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$ and Atiyah–Hirzebruch filtration 3 $cw \equiv 1\,(4)$ can be recovered from the **aAHSS**$(B_0^{\mathbb{R}}(1))$ (see Fig. 17). Notice that there can be no differentials whose target are the h_0-towers $\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2\rfloor-2}H$ for degree reasons. This determines the contributions to the E_∞-page in coweight $cw \equiv 2\,(4)$ coming from Atiyah–Hirzebruch filtration 0.

The d_1-differential between Atiyah–Hirzebruch filtrations 2 and 3 in coweight $cw \equiv 1\,(4)$ can be recovered from the **aAHSS**$(B_0^{\mathbb{R}}(1))$ (see Fig. 11). Similarly, the d_2-differential between Atiyah–Hirzebruch filtrations 0 and 2 in coweight $cw \equiv 1\,(4)$ can be recovered from the **aAHSS**$(B_0^{\mathbb{R}}(1))$ (See Fig. 14). For degree reasons, we have that $E_3 = E_\infty$ on Atiyah–Hirzebruch filtration 2, so this determines the contributions to the E_∞-page in coweight $cw \equiv 2\,(4)$ from Atiyah–Hirzebruch filtration 2.

The d_1-differential between Atiyah–Hirzebruch filtrations 2 and 3 kills all classes from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 0\,(4)$ which support h_0-multiplication. In particular, this kills all h_0-towers in the summand $\bigoplus_{j=1}^{2k} \Sigma^{4j-4,r\lfloor(j+1)/2\rfloor-4}H$ and in the big flag $\mathcal{F}_{4i,i,2i}$. The d_3-differential between Atiyah–Hirzebruch filtrations 0 and 3 is determined by the formula $d_3(\rho[3]) = (\tau h_1)[0]$ established in Lemma 7.1. In particular, using the notation of Definition 6.4, we have

$$d_3(\rho \cdot x_{4k}[3]) = (\Sigma^{2i,i}\tau h_1)[0].$$

Linearity gives the rest of the differentials. Note that this is the differential which ensures that Atiyah–Hirzebruch filtration 0 is trivial in coweight $cw \equiv 1\,(4)$ on the E_∞-page.

Thus, in coweight $cw \equiv 2\,(4)$, the E_∞-page is given by the remaining h_0-towers from the big staircase $\Sigma^{2i,i}\mathbb{S}$ from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$; the h_0-towers which do not come from the big flag

$$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2\rfloor-2}H$$

from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$; the remaining $\mathbb{F}_2[\tau^4]$'s from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 2\,(4)$; and the remaining submodule of the big flag $\mathcal{F}_{4i,i,2i}$ from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 0\,(4)$. Base change to $\mathbb{C}$ and linearity give the remaining hidden extensions, which we discuss now.

There is a hidden h_0-extension between each $\mathbb{F}_2[\tau^4]$ from Atiyah–Hirzebruch filtration 2 and the h_0-tower's concentrated in stems $s \equiv i + 4\,(8)$ from Atiyah–Hirzebruch filtration 0. Additionally, there is a hidden h_0-extension between each $\mathbb{F}_2[\tau^4]$ from Atiyah–Hirzebruch filtration 2 and the remaining submodule of the big flag $\mathcal{F}_{4i,i,2i}$ from Atiyah–Hirzebruch filtration 3. For example, there

is an h_0-extension

$$h_0 \cdot h_1 x_{4k}[3] = \Sigma^{2i,i}(h_1 \cdot \tau h_1)[2].$$

Finally, there is a hidden h_0-extension between the h_0-towers concentrated in stems $s \equiv i\,(8)$ from Atiyah–Hirzebruch filtration 0 and the remaining submodule of the big flag $\mathcal{F}_{4i,i,2i}$ from Atiyah–Hirzebruch filtration 3. For example, there is an h_0-extension

$$h_0 \cdot \rho^3 x_{4k}[3] = \Sigma^{2i,i}(\tau^2 h_0)[0].$$

By inspection, we see that the hidden extensions assemble all of the remaining classes on the E_∞-page into the big flag

$$\mathcal{F}_{4i+4,1+1,2i+2}.$$
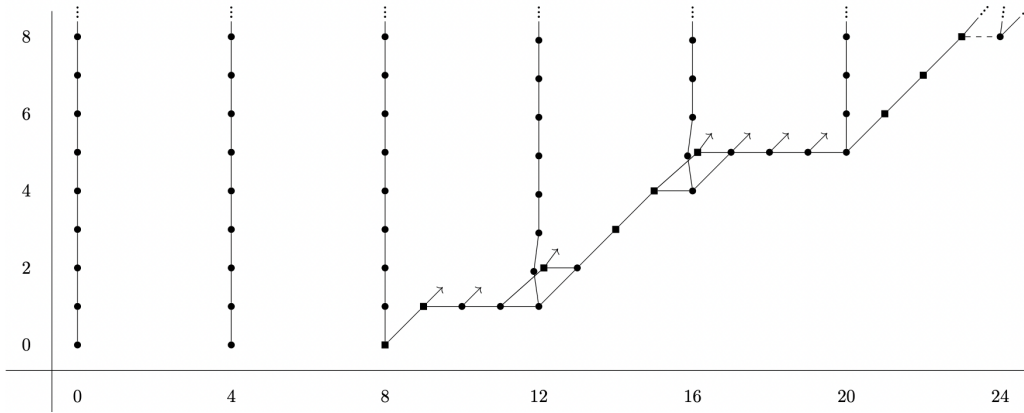
FIGURE 25. $Z_5^{\mathbb{R}}$ in $cw \equiv 2\,(4)$.

Coweight $cw \equiv 3\,(4)$

Finally, the coweight $cw \equiv 3\,(4)$ portion of the spectral sequence consists of classes coming from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 2\,(4)$ and from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 1\,(4)$. The d_1-differential between Atiyah–Hirzebruch filtrations 2 and 3 kills all classes from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 2\,(4)$ which are h_0-divisible. In particular, only the bases of the h_0-towers $\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2 \rfloor - 2} H$ survive, to the E_2-page:

$$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2 \rfloor - 2} \mathbb{F}_2[\tau^4],$$

and only bases of the h_0-towers in the big staircase $\Sigma^{2i,i}\mathcal{S}$, as well as the classes that are not h_0-divisible, survive. The d_2-differential between Atiyah–Hirzebruch filtrations 0 and 2 kills all classes from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 2\,(4)$ which support h_1-multiplication. The classes remaining from the h_0-towers that are not from the big staircase do not support h_1-multiplication, and neither do the two classes $b^n a$ and $b^n \rho a$ remaining from the staircases, so they survive to the E_3-page. For degree reasons, we have that $E_3 = E_\infty$ on Atiyah–Hirzebruch filtration 2, so this determines the contributions to the E_∞-page in coweight $cw \equiv 3\,(4)$ from Atiyah–Hirzebruch filtration 2.

The d_1-differential between Atiyah–Hirzebruch filtrations 2 and 3 kills all classes from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 1\,(4)$ which support h_0-multiplication. This can be recovered from the **aAHSS**$(B_0^{\mathbb{R}}(1))$. The d_3-differential between Atiyah–Hirzebruch filtrations 3 and 0 is only

nonzero on the bottom left corner of the remnants of the diamond. This determines the contributions to the E_∞-page in coweight $cw \equiv 3\,(4)$ from Atiyah–Hirzebruch filtration 3.

Thus, in coweight $cw \equiv 1\,(4)$, the E_∞-page is given by the segments T coming from Atiyah–Hirzebruch filtrations 2 and 3 and the bases of the h_0-towers from Atiyah–Hirzebruch filtration 2. These segments are connected by hidden extensions in the following way. Let $\Sigma^{2i,i}(h_1(\tau h_1))[3]$ and $\Sigma^{2i,i}\rho h_1(\tau h_1)[3]$ be the classes remaining from Atiyah–Hirzebruch filtration 3 in lowest Adams filtration, and let $\Sigma^{2i,i}a[2]$ and $\Sigma^{2i,i}\rho a[2]$ be the classes remaining from Atiyah–Hirzebruch filtration 2 in lowest Adams filtration. Base change to $\mathbb{C}$ gives a hidden extension

$$h_1 \cdot \Sigma^{2i,i}(h_1(\tau h_1))[3] = a[2].$$

Linearity over $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}(\mathbb{M}_2^\mathbb{R})$ gives us that

$$h_1 \cdot \Sigma^{2i,i}(\rho h_1(\tau h_1))[3] = \rho a[2]$$

and

$$h_0 \cdot \Sigma^{2i,i}(h_1(\tau h_1))[3] = h_1 \cdot \Sigma^{2i,i}(h_0(\tau h_1))[3] = h_1 \cdot \Sigma^{2i,i}(\rho h_1(\tau h_1))[3] = \rho a[2].$$

Multiplication by b gives the same extensions throughout the E_∞-page. Altogether, this gives the summand

$$\Sigma^{2i+4,i+2}\mathcal{D}\langle 1 \rangle.$$

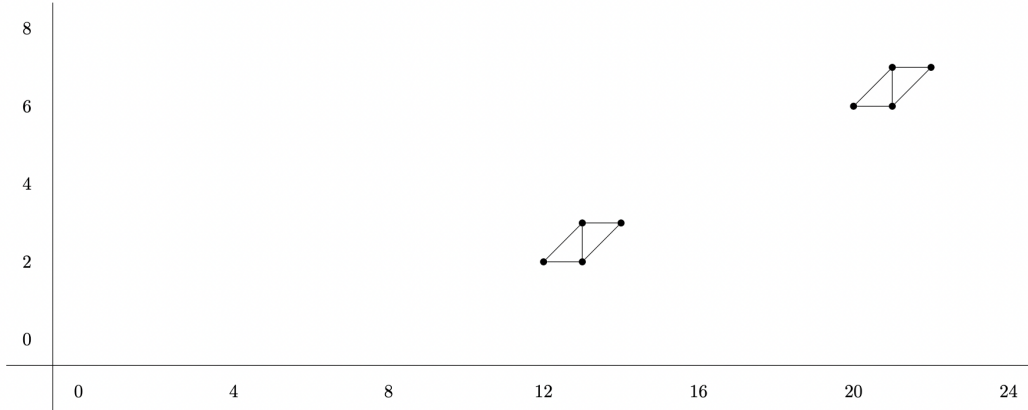For degree reasons, the bases of the h_0-towers from Atiyah–Hirzebruch filtration 2 are v_1-torsion.

FIGURE 26. $Z_5^\mathbb{R}$ in $cw \equiv 3\,(4)$.

Thus, we have shown that for $i = 4k + 1$, $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}\left(B_0^\mathbb{R}(1)^{\otimes i}\right)/v_1$-torsion is given by the direct sum of columns of Table 5. The other congruence classes are proved in a similar fashion. $\qquad\square$

Remark 7.11. It is possible that one can recover Theorem 7.10 by using the ρ-Bockstein spectral sequence and [CQ21, Lemma 3.36]. This spectral sequence has signature

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{C}^\vee}\left(B_0^\mathbb{C}(1)^{\otimes i}\right)[\rho] \implies \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}\left(B_0^\mathbb{R}(1)^{\otimes i}\right).$$

While we do not analyze this spectral sequence here, we remark that the extension problems we encounter throughout the algebraic Atiyah–Hirzebruch process are the same extension problems one encounters in the ρ-Bockstein spectral sequence.

8. The ring of cooperations

In this section, we continue the inductive procedure and compute the groups $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}(B_0^\mathbb{R}(k))$. Then, we assemble our results to compute the ring of cooperations $\pi_{**}^\mathbb{R}(\mathrm{kq} \otimes \mathrm{kq})$.

8.1. **$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(k))$.** The goal of this section is to compute $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(k))$. We begin with the following lemma.

Lemma 8.1. *There is an isomorphism of $\mathcal{A}(1)_{\mathbb{R}}^{\vee}$-comodules and $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-modules:*

$$\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k)) \cong \bigoplus_{j=0}^{k} \Sigma^{4j,2j} \mathbb{F}_2[\rho, \tau^2, h_0]/(\rho h_0).$$

Proof. Let $C_{\mathcal{A}(0)^{\vee}}(B_1^{\mathbb{R}}(k))$ denote the cobar complex computing $\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k))$. This is a cosimplicial ring which takes the form

$$B_1^{\mathbb{R}}(k) \longrightarrow \mathcal{A}(0)_{\mathbb{R}}^{\vee} \otimes B_1^{\mathbb{R}}(k) \longrightarrow \mathcal{A}(0)_{\mathbb{R}}^{\vee} \otimes \mathcal{A}(0)_{\mathbb{R}}^{\vee} \otimes B_1^{\mathbb{R}}(k) \longrightarrow \cdots$$

after totalization. Since the coaction of $\mathcal{A}(0)_{\mathbb{R}}^{\vee}$ on $B_1^{\mathbb{R}}(k)$ fixes ρ, all of the maps in $C_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k))$ are ρ-linear. This implies that the filtration by powers of ρ

$$C_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k)) \supset \rho \cdot C_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k)) \supset \rho^2 \cdot C_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k)) \supset \cdots$$

is a filtration of chain complexes. Moreover, we have the following identifications

$$\mathbb{M}_2^{\mathbb{R}}/\rho \cong \mathbb{M}_2^{\mathbb{C}}, \quad \mathcal{A}(0)_{\mathbb{R}}^{\vee}/\rho = \mathcal{A}(0)_{\mathbb{C}}^{\vee}, \quad B_1^{\mathbb{R}}(k)/\rho = B_1^{\mathbb{C}}(k).$$

Thus, the filtration on $C_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k))$ gives a ρ-Bockstein spectral sequence (see Section 6.3) of the form:

$$E_1 = \mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{C}}^{\vee}}(B_1^{\mathbb{C}}(k))[\rho] \implies \mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k)).$$

The groups $\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{C}}^{\vee}}(B_1^{\mathbb{C}}(k))$ were calculated by Culver-Quigley [CQ21, Remark 3.25]. Modulo v_1-torsion, we have

$$\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{C}}^{\vee}}(B_1^{\mathbb{C}}(k)) \cong \bigoplus_{j=0}^{k} \Sigma^{4j,2j} \mathbb{M}_2^{\mathbb{C}}[h_0].$$

In the case of $k = 0$, we have that $B_1^{\mathbb{C}}(0) \cong \mathbb{M}_2^{\mathbb{C}}$, and this ρ-Bockstein spectral sequence was completely determined by Hill [Hil11, Theorem 3.1]. We can rewrite the E_1-page as

$$E_1 = \mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}})[\rho] = \mathbb{M}_2^{\mathbb{C}}[h_0][\rho],$$

with a differential $d_1(\tau) = \rho h_0$. The Liebniz rule immediately determines the end result:

$$\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}) \cong \mathbb{F}_2[\rho, \tau^2, h_0]/(\rho h_0).$$

More generally, the E_1-page of the ρ-Bockstein takes the form

$$E_1 = \bigoplus_{j=0}^{k} \Sigma^{4j,2j} \mathbb{M}_2^{\mathbb{C}}[h_0][\rho],$$

and the differentials can be determined using the cobar complex. In particular, there are differentials

$$d_1(\Sigma^{4j,2j}\tau) = \Sigma^{4j,2j}(\rho h_0)$$

for each $0 \le j \le k$ on the classes τ in each H-tower in the decomposition of $\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{C}}^{\vee}}(B_1^{\mathbb{C}}(k))$. The Liebniz rule immediately determines the end result:

$$\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k)) \cong \bigoplus_{j=0}^{k} \Sigma^{4j,2j} \mathbb{F}_2[\rho, \tau^2, h_0]/(\rho h_0),$$

concluding the proof. $\square$

We will now use the results of Theorem 7.10 and the short exact sequences of motivic Brown–Gitler comodules to culminate our algebraic Atiyah–Hirzebruch program. The following result should be compared with [Mah81, Prop 2.6], [Beh+19, Prop 3.3], and Theorem 5.6.
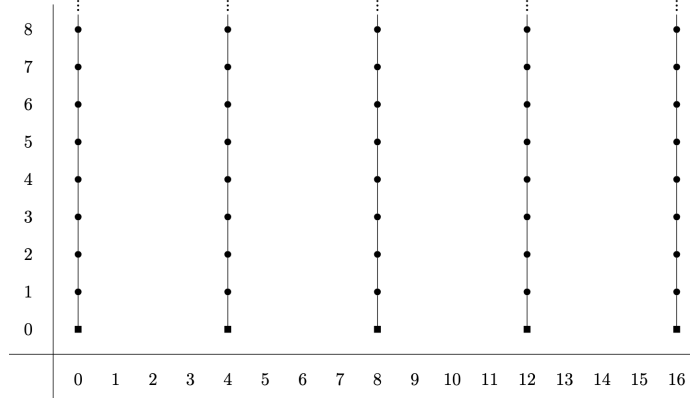
FIGURE 27. The group $\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(k))$. A black ■ denotes $\mathbb{F}_2[\rho, \tau^2]$, and a black • denotes $\mathbb{F}_2[\tau^2]$. A vertical black line represents multiplication by h_0.

Theorem 8.2. *There is an isomorphism of* $\mathcal{A}(1)_{\mathbb{R}}^{\vee}$-*comodules and* $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-*modules:*

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(k))}{v_1\text{-torsion}} \cong \Sigma^{4k-4,2k-2} Z_{\alpha(k)}^{\mathbb{R}} \oplus \bigoplus_{j=0}^{4k-8} \left(\Sigma^{4j,2j} H \oplus \Sigma^{4j,2j-2} H\right),$$

where $\alpha(k)$ *is the number of 1's in the dyadic expansion of* k. *There are* τ^2-*extensions between* $\Sigma^{4j,2j} H$ *and* $\Sigma^{4j,2j-2} H$.

Proof. We will induct on k and use the short exact sequences of Proposition 2.21. For $k = 1$, this was shown in Corollary 7.4. We assume now the theorem is true for all $i < k$ and divide the proof into two cases: when k is even, or when k is odd.

Suppose that k is even. Proposition 2.21 provides us with a short exact sequence of $\mathcal{A}(1)_{\mathbb{R}}^{\vee}$-comodules:

(8.3) $$0 \to \Sigma^{2k,k} B_0^{\mathbb{R}}(\tfrac{k}{2}) \to B_0^{\mathbb{R}}(k) \to B_1^{\mathbb{R}}(\tfrac{k}{2} - 1) \otimes (\mathcal{A}(1)_{\mathbb{R}} /\!/ \mathcal{A}(0))_{\mathbb{R}}^{\vee} \to 0.$$

Applying $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(-)$ gives a long exact sequence of $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-modules. Moreover, after killing v_1-torsion the connecting homomorphism is trivial, giving a short exact sequence whose middle term is $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(k))$. Therefore, the Ext group in question decomposes into the Ext groups of the kernel and cokernel of the short exact sequence (8.3). By the inductive hypothesis, we know that

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(\Sigma^{2k,k} B_0^{\mathbb{R}}(\tfrac{k}{2})\right) \cong \Sigma^{2k,k}\left(\Sigma^{2k-4,k-2} Z_{\alpha(k/2)}^{\mathbb{R}} \oplus \bigoplus_{j=0}^{2k-8}(\Sigma^{4j,2j} H \oplus \Sigma^{4j,2j-2} H)\right).$$

A change of rings isomorphism gives an isomorphism for the right-hand side:

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}\left(B_1^{\mathbb{R}}(\tfrac{k}{2} - 1) \otimes (\mathcal{A}(1) /\!/ \mathcal{A}(0))_{\mathbb{R}}^{\vee}\right) \cong \mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}\left(B_1^{\mathbb{R}}(\tfrac{k}{2} - 1)\right).$$

The group $\mathrm{Ext}_{\mathcal{A}(0)_{\mathbb{R}}^{\vee}}(B_1^{\mathbb{R}}(\tfrac{k}{2} - 1))$ was calculated in Lemma 8.1. Thus the right-hand side and left-hand side assemble to give

$$\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(k)) \cong \Sigma^{4k-4,2k,2} Z_{\alpha(k)}^{\mathbb{R}} \oplus \bigoplus_{j=0}^{2k-8}\left(\Sigma^{4j+2k,2j+2k} H \oplus \Sigma^{4j+4k,2j-2+2k} H\right)$$

$$\oplus \bigoplus_{j=0}^{k/2-1} \Sigma^{4j,2j} \mathbb{F}_2[\rho, \tau^2, h_0](\rho h_0),$$

using that $\alpha(\frac{k}{2}) = \alpha(k)$. Notice that for degree reasons, any ρ-divisible class coming from $\mathrm{Ext}_{\mathcal{A}(0)_\mathbb{R}^\vee}(B_1^\mathbb{R}(\frac{k}{2}-1))$ is v_1-torsion. Combining and reindexing the H-towers in the second and third summands gives the result.

Suppose now that k is odd. Proposition 2.21 provides us with a different short exact sequence of $\mathcal{A}(1)_\mathbb{R}^\vee$-comodules:

$$(8.4) \qquad 0 \to \Sigma^{2(k-1),k-1}B_0^\mathbb{R}(\tfrac{k-1}{2}) \otimes B_0^\mathbb{R}(1) \to B_0^\mathbb{R}(k) \to B_1^\mathbb{R}(\tfrac{k-1}{2}-1) \otimes (\mathcal{A}(1) /\!/ \mathcal{A}(0))_\mathbb{R}^\vee \to 0.$$

Again, modulo v_1-torsion the connecting homomorphism in the long exact sequence obtained by applying $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}(-)$ is trivial, giving a short exact sequence whose middle term is $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}(B_0^\mathbb{R}(k))$. We proceed by analyzing the Ext groups of the kernel and cokernel of the short exact sequence (8.4). A change of rings isomorphism gives the right-hand side:

$$\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}\left(B_1^\mathbb{R}(\tfrac{k-1}{2}-1) \otimes (\mathcal{A}(1) /\!/ \mathcal{A}(0))_\mathbb{R}^\vee\right) \cong \mathrm{Ext}_{\mathcal{A}(0)_\mathbb{R}^\vee}\left(B_1^\mathbb{R}(\tfrac{k-1}{2}-1)\right).$$

The group $\mathrm{Ext}_{\mathcal{A}(0)_\mathbb{R}^\vee}\left(B_1^\mathbb{R}(\tfrac{k-1}{2}-1)\right)$ was calculated in Lemma 8.1. To calculate the left-hand side, we can use the **aAHSS** $\left(B_0^\mathbb{R}(\tfrac{k-1}{2}) \otimes B_0^\mathbb{R}(1)\right)$. This spectral sequence is obtained by applying the functor $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}\left(B_0^\mathbb{R}(\tfrac{k-1}{2}) \otimes -\right)$ to the cellular filtration for $B_0^\mathbb{R}(1)$ and has signature

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}\left(B_0^\mathbb{R}(\tfrac{k-1}{2})\right) \otimes \mathbb{M}_2^\mathbb{R}\{[1], [\bar{\xi}_1], [\bar{\tau}_1]\} \implies \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}\left(B_0^\mathbb{R}(\tfrac{k-1}{2}) \otimes B_0^\mathbb{R}(1)\right).$$

By the inductive hypothesis, we have an isomorphism modulo v_1-torsion

$$(8.5) \qquad \mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}\left(B_0^\mathbb{R}(\tfrac{k-1}{2})\right) \cong \Sigma^{2k-6,k-3}Z_{\alpha(k)-1}^\mathbb{R} \oplus \bigoplus_{j=0}^{2k-10}\left(\Sigma^{4j,2j}H \oplus \Sigma^{4j,2j-2}H\right),$$

using that $\alpha(\frac{k-1}{2}) = \alpha(k) - 1$. This splitting descends to a splitting of the spectral sequence, and we may analyze each summand of (8.5) individually. The left-hand summand is handled by Theorem 7.10, as it is isomorphic to the **aAHSS** $\left(\Sigma^{2k-6,k-3}B_0^\mathbb{R}(1)^{\otimes\alpha(k)}\right)$, so we have a summand of $\Sigma^{2k-2,k-1}Z_{\alpha(k)}^\mathbb{R}$ in $\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}\left(B_0^\mathbb{R}(\tfrac{k-1}{2}) \otimes B_0^\mathbb{R}(1)\right)$. The algebraic Atiyah–Hirzebruch spectral sequence on the right-hand summand collases on E_2. The remaining classes in Adams filtration 0 are v_1-torsion, and so the result is isomorphic to the original right-hand summand. Thus, we have an isomorphism:

$$\Sigma^{2(k-1),k-1}\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}\left(B_0^\mathbb{R}(\tfrac{k-1}{2}) \otimes B_0^\mathbb{R}(1)\right)$$

$$\cong \Sigma^{2(k-1),k-1}\left(\Sigma^{2k-2,k-1}Z_{\alpha(k)}^\mathbb{R} \oplus \bigoplus_{j=0}^{2k-10}\left(\Sigma^{4j,2j}H \oplus \Sigma^{4j,2j-2}H\right)\right).$$

Assembling the right-hand side and left-hand side gives

$$\mathrm{Ext}_{\mathcal{A}(1)_\mathbb{R}^\vee}(B_0^\mathbb{R}(k)) \cong \Sigma^{4k-4,2k-2}Z_{\alpha(k)}^\mathbb{R} \oplus \bigoplus_{j=0}^{2k-10}\left(\Sigma^{4j+2(k-1),2j+k-1}H \oplus \Sigma^{4j+2(k-1),2j-2+k-1}H\right)$$

$$\oplus \bigoplus_{j=0}^{(k-1)/2-1}\Sigma^{4j,2j}\mathbb{F}_2[\rho,\tau^2,h_0]/(\rho h_0).$$

As before, for degree reasons, any ρ-divisible class coming from $\mathrm{Ext}_{\mathcal{A}(0)_\mathbb{R}^\vee}\left(B_1^\mathbb{R}\left(\tfrac{k-1}{2}-1\right)\right)$ is v_1-torsion. Combining and reindexing the H-towers in the second and third summands gives the result. $\square$

Remark 8.6. The summand $\bigoplus_{j=0}^{4k-8}\left(\Sigma^{4j,2j}H \oplus \Sigma^{4j,2j-2}H\right)$ appearing in this decomposition is a ghost of the failure to express these Ext groups in terms of Adams covers of kq and ksp. It was shown in [Hil11] that the motivic Adams spectral sequence for $H\mathbb{Z}$ collapses at the E_2-page and has signature

$$E_2 = \mathrm{Ext}_{\mathcal{A}(0)_\mathbb{R}^\vee}(\mathbb{M}_2^\mathbb{R}) \cong \mathbb{F}_2[\rho,\tau^2,h_0]/(\rho h_0) \implies \pi_{**}^\mathbb{R}(H\mathbb{Z}).$$

In the Adams covers for kq and ksp, one sees summands of $\pi_{**}^\mathbb{R}H\mathbb{Z}$. In the context of our calculations, we only obtain summands of $\pi_{**}^\mathbb{R}(H\mathbb{Z}/\rho)$.

8.2. **The ring of cooperations.** In this subsection only, all results are implicitly computed modulo v_1-torsion. We assemble our results to compute the ring of cooperations $\pi_{**}^{\mathbb{R}}(\mathrm{kq} \otimes \mathrm{kq})$. First, our computation in Theorem 8.2 gives the following:

Corollary 8.7. *The E_2-page of the* **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$ *is given by*

$$E_2 = \bigoplus_{k \geq 0} \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}^{s,f,w}(\Sigma^{4k,2k} B_0^{\mathbb{R}}(k)) \cong \bigoplus_{k \geq 0} \left(\Sigma^{4k-4,2k-2} Z_{\alpha(k)}^{\mathbb{R}} \oplus \bigoplus_{j=0}^{4k-8} \left(\Sigma^{4j,2j} H \oplus \Sigma^{4j,2j-2} H \right) \right).$$

Proof. The left hand equality, which is true even in the presence of v_1-torsion, is the content of Theorem 4.1. The right hand isomorphism follows from Theorem 8.2. $\square$

We now arrive at our main result.

Theorem 8.8. *The* **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$ *collapses on the E_2-page.*

Proof. There is a base change functor

$$- \otimes \mathbb{C} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(\mathbb{C})$$

which induces a highly structured morphism with the $\mathbb{C}$-motivic Adams spectral sequence for $\pi_{**}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$. In particular, [CQ21, Corollary 3.43] implies that there can be no differentials in the **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$ with both the source and target being τ-free. However, by Corollary 8.7 we see that every class on the E_2-page is τ^4-periodic. Hence there can be no differentials and the spectral sequence collapses.. $\square$

Remark 8.9. One may now use the ideas of Remark 4.8 to study the kq-resolution one filtration at a time. The n-line of the kq-resolution may be computed by a motivic Adams spectral sequence taking the form

$$E_2 = \bigoplus_{I \in \mathcal{I}_n} \Sigma^{4|I|,2|I|} \mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(B_0^{\mathbb{R}}(I)) \implies \pi_{*+n,*}^{\mathbb{R}}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}).$$

This spectral sequences collapses for the same reasons given above. Note that although there is an algebra structure on the E_1-page, the individual n-lines do not carry any ring structure, as $\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}$ is not a ring spectrum. These computation do present the n-line as a module over the 0-line, given by $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$.

Determining the differentials in the kq-resolution is much more difficult. By complex Betti realization, we can deduce much of the behavior of the complex kq-resolution from the bo-resolution (see Corollary 2.4). However, there is information in the real kq-resolution which is both ρ-torsion and v_1-periodic that is not detected by these methods. We plan to analyze the real kq-resolution in future work using a combination of these methods and C_2-equivariant homotopy theory.

Appendix A. Charts

In this appendix, we record charts depicting the $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-modules described in Section 6. A black $\blacksquare$ represents $\mathbb{F}_2[\tau^4][\rho, h_1]$. A black $\bullet$ represents $\mathbb{F}_2[\tau^4]$. A vertical black line represents multiplication by h_0. A horizontal black line represents multiplication by ρ. A diagonal black line represents multiplication by h_1.
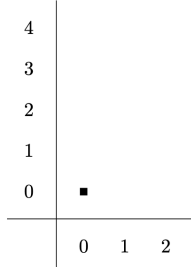

FIGURE 28. The $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module P.
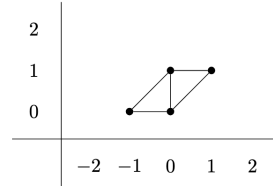
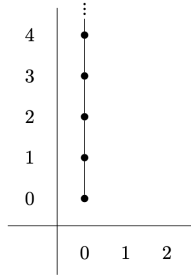FIGURE 31. The $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module D.

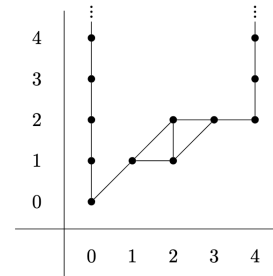FIGURE 29. The $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module H.

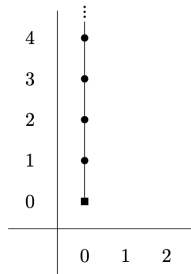FIGURE 32. The $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module S.

FIGURE 30. The $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module PH.

FIGURE 33. The $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module T.

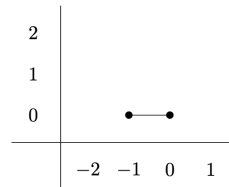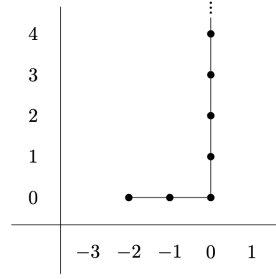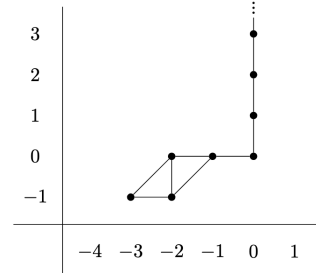FIGURE 34. The $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module J.

FIGURE 35. The $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{R}}^{\vee}}(\mathbb{M}_2^{\mathbb{R}})$-module JD.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON

Email address: jackmann@uw.edu