THE ADAMS SPECTRAL SEQUENCE AND HOPF ALGEBROIDS

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Abstract. In this talk, we briefly construct the Adams spectral sequence and discuss its connection with Hopf algebroids.

Contents

1. Recollections

Let's recall what we've done in the seminar so far. First, we introduced **complex-oriented cohomology theories**. These are multiplicative cohomology theories (or ring spectra) E which admit Thom classes to every complex vector bundle. These cohomology theories have certain classes $x_E \in E^2(\mathbb{CP}^\infty)$ which restrict to a generator of $\pi_0 E$ induced by the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$. Such a class is called a **complex orientation** for E. With this class, we are able to construct a formal group law over E_*. There are universal complex vector bundles $\gamma_n \to BU(n)$ over the Grassmanian classifying rank n complex vector bundles, and the Thom spaces $MU(n)$ of these bundles assemble into the complex cobordism spectrum MU. This is the universal complex-oriented cohomology theory in the sense that a complex orientation of E is equivalent to the data of a ring map $MU \to E$, and any such map will classify the formal group law over E_*. This last fact is nontrivial, and requires a delicate analysis. One way to this fact is via the Adams spectral sequence, a homological machine which takes spectra as input and has homotopy groups as output. We will investigate this spectral sequence before taking a detour into the theory of Hopf algebroids. In this talk, we will not assume that E is complex-oriented.

2. Adams spectral sequence I

Let E be a ring spectrum. We have two structure maps,

$$i : \mathbb{S} \to E, \quad m : E \otimes E \to E,$$

satisfying left and right unitality and associativity. We want to find a way to "do homological algebra" over E, and we can basically do what this.

Definition 2.1. A sequence of spectra

$$A_1 \to A_2 \to \cdots A_n$$

is ***E*-exact** if after smashing with E, the sequence

$$A_1 \otimes E \to A_2 \otimes E \to \cdots \to A_n \otimes E$$

is exact *as homotopy functors*. We note that the homotopy functor associated to a spectrum X is given by $[X, -]$. The above condition then translates to the sequence of functors

$$[A_n \otimes E, -] \to \cdots \to [A_2 \otimes E, -] \to [A_1 \otimes E, -]$$

being exact as functors into abelian groups. We will say that a map of spectra $f : A \to B$ is **E-monic** if the sequence $* \to A \to B$ is E-exact. Finally, we call a spectrum I an **E-injective** of for any E-monic map $f : A \to B$ and any map $g : A \to I$, there is a lift up to homotopy $h : B \to I$ making the expected diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "h"] \\ I & \end{tikzcd}$$

With this homological algebra in place, we are able to make our first construction.

Definition 2.2. An **E-Adams resolution** of a spectrum X is an E-exact sequence

$$* \to X \xrightarrow{j_0} I_0 \xrightarrow{j_1} I_1 \to \cdots$$

such that:

- $j_n \circ j_{n-1} \simeq 0$;
- I_n is E-injective.

This is the analogue of an injective resolution that we will work with, and we will show that one always exists by explicit construction. In fact, we will construct two!

2.1. **Normalized E-Adams resolution.** The first resolution we will construct is dubbed the **normalized E-Adams resolution**. This is a resolution that comes from iteratively extending the unit map on our ring spectrum E, then smashing with our spectrum X. It's not the best for computational purposes; for example, if X is also a ring spectrum, then one would like the maps in this resolution to be ring spectrum maps, and this is not the case. However, it is easy to show that this is indeed a resolution, and so we start with this one.

Take the unit map on E and extend to a cofiber sequence:

$$\mathbb{S} \to E \to \overline{E}.$$

We can slide another copy of the sphere spectrum onto the cofiber $\overline{E}$ and apply the unit map again, giving a sequence

$$\mathbb{S} \otimes \overline{E} \to E \otimes \overline{E} \to \overline{E} \otimes \overline{E}.$$

Note that by iterating this process, we can construct a sequence

$$\begin{tikzcd} \mathbb{S} \arrow[r] & E \arrow[r, dashed] \arrow[d] & E \otimes \overline{E} \arrow[r, dashed] \arrow[d] & E \otimes \overline{E} \otimes \overline{E} \arrow[r, dashed] \arrow[d] & \cdots \\ & \overline{E} \arrow[ur] & \overline{E} \otimes E \arrow[ur] & \overline{E} \otimes \overline{E} \otimes E \arrow[ur] & \end{tikzcd}.$$

It is this top sequence which is our normalized E-Adams resolution. Smashing with X, we get a sequence

$$X \to E \otimes X \to E \otimes \overline{E} \otimes X \to E \otimes \overline{E} \otimes \overline{E} \otimes X \to \cdots$$

Although it is clear by construction that each consecutive map is nullhomotopic, we need to do a little bit of work to show that this is indeed an Adams resolution.

Lemma 2.1. *For any X, the map $X \to E \otimes X$ induced by the unit map on E is E-monic.*

Proof. We need to show that the sequence $* \to [E \otimes X, -] \to [E \otimes E \otimes X, -]$ is exact, meaning that for any Y, there is a surjection $[E \otimes E \otimes X, Y] \to [E \otimes X, Y]$. Note that the surjection in question is given by precomposition with the unit map for E, and that multiplication on E gives a natural map

$$E \otimes E \otimes X \to E \otimes X.$$

This gives us our answer. For any $f : E \otimes X \to Y$, we see that $f \circ m : E \otimes E \otimes X \to E \otimes X \to Y$ is the preimage under the unit map, since $f \circ m \circ i = f \circ \mathrm{id} = f$ by unitality of E. □

This lemma also shows us that $E \otimes X$ is a retract of $E \otimes E \otimes X$. Additionally, this shows that if $r : A \to B$ is any map with a retraction $s : B \to A$, then r is monic. By moving around the cofiber sequence induced by the unit map on E, it is not hard to see that each of the maps in the normalized E-Adams resolution must be E-monic. However, we still need to show that they are E-injective

Lemma 2.2. *A spectrum I is E-injective if and only if I is a retract of $E \otimes I$.*

Proof. Suppose I is E-injective. We want to show that I is a retract of $E \otimes I$, which is equivalent to showing that there is a lift in the following diagram:

$$\begin{tikzcd} I \arrow[r, "i\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & E \otimes I \arrow[dl, dashed] \\ I & \end{tikzcd}$$

Since I is E-injective, we get a lift if we can show that $I \to E \otimes I$ is E-monic. This is Lemma 1. Now, suppose that $I \to E \otimes I$ includes as a retract, that $f : A \to B$ is an E-monomorphism, and $g : A \to I$ is any map. We must provide a lift h such that $hf = g$. There is a cofiber sequence

$$E \otimes A \xrightarrow{\mathrm{id}\otimes f} E \otimes B \to C \to \Sigma(E \otimes A).$$

Since $f : A \to B$ is E-monic, the map $C \to \Sigma(E \otimes A)$ must be 0, hence factors through $*$. Now, if we look at the identity map $E \otimes I \to E \otimes I$, we see that it has cofiber $*$. Consider the following diagram:

$$\begin{tikzcd} E \otimes A \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "\mathrm{id}\otimes g"'] & E \otimes B \arrow[r] \arrow[d, dashed, "\tilde{h}"] & C \arrow[r] \arrow[d] & \Sigma(E \otimes A) \arrow[d] \\ E \otimes I \arrow[r] & E \otimes I \arrow[r] & * \arrow[r] & \Sigma(E \otimes I) \end{tikzcd}$$

We have a commutative diagram on the rightmost square by our argument before, and so by the triangulated category axioms we have the dashed map $\tilde{h} : E \otimes B \to E \otimes I$.Let $r : E \otimes I \to I$ be the map witnessing the retraction. We claim now that $h = r \circ \tilde{h} \circ (e \otimes \mathrm{id})$ is a lift of g. To see this, observe:

$$\begin{aligned} hf &= (r \circ \tilde{h} \circ (e \otimes \mathrm{id})) \circ f \\ &= (r \circ \tilde{h} \circ (\mathrm{id} \otimes f)) \circ (e \otimes \mathrm{id}) \\ &= (r \circ (\mathrm{id} \otimes g)) \circ (e \otimes \mathrm{id}) \\ &= (r \circ e \otimes \mathrm{id}) \circ g = g \end{aligned}$$

The first equality is definition. The second is clear. The third comes from the left hand square in the diagram which constructs $\tilde{h}$. The fourth is clear. This gives a lift and we are done. We organize the last argument in the following commutative diagram.

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr, "g"] \arrow[ddd, "e\otimes\mathrm{id}"'] & & & B \arrow[ddd, "e\otimes\mathrm{id}"] \\
& I \arrow[dr, bend right, "e\otimes\mathrm{id}"'] & & \\
& & E\otimes I \arrow[ul, bend right, "r"'] & \\
E\otimes A \arrow[urr, "\mathrm{id}\otimes g"'] \arrow[rrr, "\mathrm{id}\otimes f"'] & & & E\otimes B \arrow[ul, "\tilde{h}"']
\end{tikzcd}$$

□

This shows the following.

Corollary 2.1. *There are enough E-injectives in the category of spectra.*

In some sense, we can do homological algebra with spectra by applying π_* everywhere. However, we want to work with a better resolution than the one we constructed.

2.2. **Bar complex.** To any ring spectrum E, we can associate a cosimplicial spectrum $E^\bullet$ just by using the unit and multiplication maps:

$$\begin{tikzcd}
\mathbb{S} \arrow[r] & E \arrow[r, shift left=2] \arrow[r, shift right=2] & E\otimes E \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & E\otimes E\otimes E \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[r] & \cdots
\end{tikzcd}$$

For instance there are two maps $E \to E \otimes E$ given by $\eta_L := (e \otimes \mathrm{id})$ and $\eta_R := (\mathrm{id} \otimes e)$, and there are two maps $E \otimes E \otimes E \to E \otimes E$ given by $(m \otimes \mathrm{id})$ and $(\mathrm{id} \otimes m)$. To any cosimplicial object we have its **totalization**, the dual to the geometric realization of a simplicial object. Totalization forms a diagram with the same objects as $E^\bullet$ and but with a single arrow $E^{\otimes n} \to E^{\otimes(n+1)}$ given by an alternating sum of the codegeneracy maps.

$$\mathbb{S} \longrightarrow E \xrightarrow{\ \delta\ } E \otimes E \xrightarrow{\ \delta\ } E \otimes E \otimes E \xrightarrow{\ \delta\ } \cdots$$

For instance, $\delta : E \to E \otimes E$ is given by $\eta_L - \eta_R$. It is clear that if X is any spectrum, then smashing this diagram with X yields a diagram

$$X \longrightarrow E \otimes X \xrightarrow{\ \delta\ } E \otimes E \otimes X \xrightarrow{\ \delta\ } E \otimes E \otimes E \otimes X \xrightarrow{\ \delta\ } \cdots$$

This is the **standard resolution** of X. We will take as fact that this is indeed an E-Adams resolution of X (it is not hard to show given what we've done already). Another thing we won't show is the following analogue from homological algebra.

Lemma 2.3. *Suppose we have E-Adams resolutions*

$$* \to X \to I_0 \to I_1 \to \cdots$$

$$* \to Y \to J_0 \to J_1 \to \cdots .$$

If $f : X \to Y$ is any map, then there is a map of resolutions $I_\bullet \to J_\bullet$ lifting f which is unique up to chain homotopy.

In other words, we can fill in the following diagram:

$$\begin{tikzcd}
* \arrow[r] & X \arrow[r] \arrow[d, "f"'] & I_0 \arrow[r] \arrow[d, dashed] & I_1 \arrow[r] \arrow[d, dashed] & \cdots \\
* \arrow[r] & Y \arrow[r] & J_0 \arrow[r] & J_1 \arrow[r] & \cdots
\end{tikzcd}$$

For computation, this is extremely beneficial! This tells us that any two E-Adams resolutions of X are chain homotopic. Thus, if we want to compute derived functors, we are free to choose whichever resolution suits us best.

2.3. **E-Adams tower.** We have seen how to construct a spectral analogue of injective resolutions, which is great and all, but it doesn't tell us how to form a spectral sequence yet. We do this now. An **E-Adams tower** for a spectrum X is the data of the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X_2 \arrow[r] \arrow[d] & \Sigma^{-2} I_3 \\
 & X_1 \arrow[r] \arrow[d] & \Sigma^{-1} I_2 \arrow[ul, dashed] \\
X \arrow[r, "j_0"] \arrow[ur] \arrow[uur] & X_0 = I_0 \arrow[r] & I_1 \arrow[ul, dashed]
\end{tikzcd},$$

where each hook

$$X_{n+1} \to X_n \to \Sigma^{-n} I_{n+1}$$

is a cofiber sequence, and the composites

$$j_{n+1} : I_n \to \Sigma^n X_n \to I_{n+1}$$

give rise to an E-Adams resolution

$$* \to X \to I_0 \to I_1 \to \cdots .$$

Moreover, if we have an E-Adams resolution of X, then we can build an E-Adams tower. In more details, suppose we have an E-Adams resolution $* \to X \to I_0 \to I_1 \to I_2 \to \cdots$. We can take $X_0 = I_0$, and are forced then to take $X_1 = \mathrm{fib}(I_0 \to I_1)$. This gives us the diagram:

$$\begin{tikzcd}
 & X_1 \arrow[d] & & \\
X \arrow[r] \arrow[ur, dashed, "h"] & I_0 \arrow[r] & I_1 \arrow[r] \arrow[d] & I_2 \\
 & & \Sigma X_1 \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

We have a map $h : X \to X_1$ since the composite $X \to I_0 \to I_1$ is nullhomotopic, and we have a map $g : \Sigma X_1 \to I_2$ since the composite $I_0 \to I_1 \to I_2$ is nullhomotopic. Desuspending gives a map $\Sigma^{-1} : X_1 \to \Sigma^{-1} I_2$. We can assemble into a more familiar diagram now:

$$\begin{tikzcd}
 & X_1 \arrow[r, "\Sigma^{-1} g"] \arrow[d] & \Sigma^{-1} I_2 \\
X \arrow[r] \arrow[ur, "h"] & I_0 \arrow[r] & I_1
\end{tikzcd}$$

We are forced here to take $X_2 = \mathrm{fib}(\Sigma^{-1} g : X_1 \to \Sigma^{-1} I_2)$, and it is in this way that we produce the tower. This constructs the cofiber sequences as hooks. Moreover, the map $I_1 \to \Sigma X_1 \to I_2$ given by the connecting map and g recovers the map $I_1 \to I_2$ from the original E-Adams resolution. Thus, since any spectrum has an E-Adams resolution, we also have that every spectrum has an E-Adams tower.

2.4. **E-Adams spectral sequence.** The point here is that an $E-$Adams tower gives us a spectral sequence after applying π_*, and the E_1-page of this spectral sequence is the corresponding E-Adams resolution. This is the **E-Adams spectral sequence**. By Lemma 2.2.1, this process is independent of choice of spectral sequence, and so we may identify the E_1-page as the complex:

$$\pi_*(E\otimes X) \rightleftarrows \pi_*(E\otimes E\otimes X) \rightleftarrows \pi_*(E\otimes E\otimes E\otimes X) \longrightarrow \cdots$$

Remark 2.3. We will not mention convergence much here, but we will say that when X has bounded below homotopy groups, this spectral sequence converges to $\pi_* X^\wedge_E$, the **E-nilpotent completion** of X.

This doesn't look very nice, and in fact it usually isn't very nice. We've just given ourselves a bunch of different homotopy groups to compute, which was kinda what were trying to sidestep by even making a spectral sequence in the first place. Of course, if such a spectral sequence were so incomputable then we wouldn't really be talking about it here, and indeed we will take some steps to make our lives easier. For notational convenience, let $E_* := \pi_* E$ and $E_* X := \pi_*(E\otimes X)$.

We have taken the assumption that E is a ring spectrum, which gives us two honest graded rings to work with: E_* and E_*E. Notice that if we take $X = \mathbb{S}$, these constitute the first two terms in the E_1-page in the E-Adams spectral sequence for $\mathbb{S}$. There are two maps in the E_1-page of interest:

$$E_* \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} E_*E \ ,$$

coming from the maps $\eta_L : \mathbb{S}\otimes E \to E\otimes E$ and $\eta_R : E\otimes\mathbb{S} \to E\otimes E$. We will call these the **left/right units**, and we observe that they differ by the swap map $E\otimes E \to E\otimes E$. **From now on, we will work under the assumption that η_L is flat as a map of rings.** We will call E a **flat** ring spectrum in this case.

Remark 2.4. This is indeed an assumption. There are many spectra of interest that are flat, such as $H\mathbb{F}_p$, MU, BP, and any Landweber exact ring spectrum, but there are many more that are not, such as ko, tmf, and really any ring spectrum that you would generically pick out of a bag. Flatness is rare, but it makes computations with the Adams spectral sequence much easier, and so we are prone to seeing flat ring spectra in more abundance than they actually appear.

Flatness will help us understand the E_1-page of the E-Adams spectral sequence and move to the E_2-page. Note that for any spectra A and B, there is a natural map $\pi_*(A)\otimes\pi_*(B) \to \pi_*(A\otimes B)$. Thinking in terms of E_*, we can look use this observation and the universal property of the tensor product as a coequalizer to produce the following diagram:

$$\begin{tikzcd}
E_*E\otimes E_*\otimes E_*X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & E_*E\otimes E_*X \arrow[r] \arrow[d] & E_*E\otimes_{E_*} E_*X \arrow[d, dashed] \\
\pi_*(E\otimes E\otimes E\otimes E\otimes X) \arrow[r, shift left] \arrow[r, shift right] & \pi_*(E\otimes E\otimes E\otimes X) \arrow[r] & \pi_*(E\otimes E\otimes X)
\end{tikzcd}$$

The vertical maps are the maps described at the beginning of this paragraph. The topmost horizontal maps are the two action maps of E_* on E_*E and E_*X and passing to the tensor product, and the bottommost horizontal maps come from multiplying the second and third copy of E and the third and fourth copy of E, then multiplying the middle two factors of E.

Proposition 2.1. *If E is flat, then the natural map*

$$E_*E\otimes_{E_*} E_*X \to \pi_*(E\otimes E\otimes X)$$

is an isomorphism.

Proof. Treating X as a variable, we can view both sides of this map as a homology theory on spectra: certainly the target is the homology theory associated to $E\otimes E$, but flatness ensures that $E_*E\otimes_{E_*} E_*(-)$ sends cofibration sequences to long exact sequences, thus is also a homology theory. They agree on the sphere by construction, and both sides send infinite wedges to infinite direct sums, so they are indeed the same homology theories. □

This allows for a huge simplification of the E-Adams spectral sequence. We have written the third term of the E_1-page, $\pi_*(E \otimes E \otimes X)$, completely in terms of the second term and the homology of X. In particular, this tells us that *the entire Adams E_1-page is determined by E_*, E_*E, and E_*X.* Using our proposition, we can rewrite the E_1-page as

$$E_*X \to E_*E \otimes_{E_*} E_*X \to E_*E \otimes_{E_*} \otimes E_*E \otimes_{E_*} E_*X \to \cdots$$

Certainly the homological algebra is easier. However, there is more we can say!

3. Hopf algebroids

In the last section, we saw how we can use homotopy groups to fabricate some homological algebra, leading to a spectral sequence with an identifiable E_1-page. Even more, we saw that we can understand this E_1-page as a particular injective resolution involving only a finite amount of information, namely E_* and E_*E. What we will show is that this pair (E_*, E_*E) has much more structure than just a pair of rings.

3.1. **Groups, cogroups, and representable functors.** Recall that a categorical description of a group is a set G with morphisms $m : G \times G \to G$, $e : * \to G$, and $i : G \to G$ representing multiplication, unit, and inverses. These maps satisfy commutative diagrams witnessing associativity, unitality, and the inverse:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \times * = G = * \times G \arrow[r, "e \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times e"'] \arrow[dr, "\simeq"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \times G \arrow[rr, "i \times \mathrm{id}"] & & G \times G \arrow[d, "m"] \\ G \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r] & * \arrow[r, "e"] & G \\ G \times G \arrow[rr, "\mathrm{id} \times i"'] & & G \times G \arrow[u, "m"'] \end{tikzcd}$$

These diagrams allow us to generalize.

Definition 3.1. A **group object** in a category $\mathcal{C}$ with finite products is an object G with morphisms $m : G \times G \to G$, $e : 1 \to G$, and $i : G \to G$ satisfying the above commutative diagrams, where 1 is the terminal object.

A group object is **commutative** if the product commutes with the swap map, given by the following commutative diagram:

$$\begin{tikzcd} G \times G \arrow[r, "\mathrm{swap}"] \arrow[dr, "m"'] & G \times G \arrow[d, "m"] \\ & G \end{tikzcd}$$

A group object in <u>Set</u> is just a group. A group object in <u>Grp</u> is actually an abelian group: if the inverse map for G is a group homomorphism, then one can show that G must be abelian. This abstract definition allows us to define groups in more generality and see interesting consequences. It also allows us to form a "dual" notion that otherwise may have seemed contrived.

Definition 3.2. A **cogroup object** in a category C with finite coproducts is an object X with morphisms $\Delta : X \to X \coprod X$, $\varepsilon : X \to 0$, and $c : X \to X$ representing comultiplication, counit, and coinverse (or conjugation), where 0 is the terminal object. These morphisms satisfy diagrams dual to those for a group object, witnessing coassociativity:

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \coprod X \arrow[d, "\mathrm{id} \coprod \Delta"] \\
X \coprod X \arrow[r, "\Delta \coprod \mathrm{id}"] & X \coprod X \coprod X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\cong"] & X \coprod X \arrow[d, "\varepsilon \coprod \mathrm{id}"] \\
X \coprod X \arrow[r, "\mathrm{id} \coprod \varepsilon"] & X \coprod 0 \cong X \cong 0 \coprod X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \coprod X \arrow[rr, "c \coprod \mathrm{id}"] & & X \coprod X \arrow[d, "\nabla"] \\
X \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r, "\varepsilon"] & 0 \arrow[r] & X \\
X \coprod X \arrow[rr, "\mathrm{id} \coprod c"] & & X \coprod X \arrow[u, "\nabla"']
\end{tikzcd}$$

A cogroup object is **cocommutative** if the comultiplication commutes with the swap map, given by the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & X \coprod X \arrow[d, "\mathrm{swap}"] \\
& X \coprod X
\end{tikzcd}$$

In fact, cogroup appear pretty frequently. In $\underline{\mathrm{Top}_*}$, the suspension of any space ΣX is a cogroup object, where $\Delta : \Sigma X \to \Sigma X \coprod \Sigma X$ is given by the pinch map: if we pinch the suspension of a space at the equator, then we are left with two suspensions with basepoint the pinch. In this way, the spheres S^n for $n \geq 1$ are all cogroup objects in $\underline{\mathrm{hTop}_*}$, and for $n \geq 2$ they are cocommutative (this is why homotopy groups $\pi_n X$ are abelian for $n \geq 2$!). Our primary motivation for defining cogroup objects is the following example: if k is a commutative ring, then a cogroup object in $\underline{k\text{-Alg}}$ is a **Hopf algebra**. This is the structure of a k-module H that has a product and unit map (since it is an algebra), with an additional coproduct, counit map, and coinverse. Note that the cogroup structure maps are forced to be algebra maps, and that the coproduct in $\underline{k\text{-Alg}}$ is the tensor product.

There is a nice way to relate group objects and cogroup objects together. We will assume now that our category $\mathcal{C}$ is locally small, meaning that $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a set for every pair of objects $X, Y \in \mathcal{C}$. In this case, the Yoneda lemma tells us that each object $X \in \mathcal{C}$ is determines and is determined by the functor it represents, i.e. the functor $\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \to \underline{\mathrm{Set}}$. We will call any functor of the form $\mathrm{Hom}_{\mathcal{C}}(-, X)$ a **representable functor**. More concretely, the assignment $X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$ gives a full and faithful embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \underline{\mathrm{Set}})$ of $\mathcal{C}$ into its presheaf category. We may as well study these functors instead of the objects of $\mathcal{C}$ then, particularly if $\mathcal{C}$ is difficult to understand. How does the notion of a group object translate to this perspective? Well, if G is a group object in $\mathcal{C}$, then the multiplication translates to a natural transformation of representable functors:

$$m : \mathrm{Hom}_{\mathcal{C}}(-, X) \times \mathrm{Hom}_{\mathcal{C}}(-X) \to \mathrm{Hom}_{\mathcal{C}}(-, X).$$

In particular, if Y is any other object in $\mathcal{C}$, then we have the following map:

$$m_Y : \mathrm{Hom}_{\mathcal{C}}(Y, X) \times \mathrm{Hom}_{\mathcal{C}}(Y, X) = \mathrm{Hom}_{\mathcal{C}}(Y \times Y, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

This gives a multiplication on the set $\mathrm{Hom}_{\mathcal{C}}(Y, X)$! We can translate the other structure maps and the commutative diagrams for these maps to obtain the following.

Definition 3.3. A group object in a locally small category $\mathcal{C}$ is an object G such that the set $\mathrm{Hom}_{\mathcal{C}}(Y, G)$ has a natural group structure for each object Y. In other words, there is a lift of the representable functor:

$$\begin{tikzcd}
& \underline{\mathrm{Grp}} \arrow[d, "U"] \\
\mathcal{C}^{op} \arrow[r] \arrow[ur, dashed, "\mathrm{Hom}_{\mathcal{C}}(-,G)"] & \underline{\mathrm{Set}}
\end{tikzcd}$$

By virtue of "throwing co- in front of everything", there is a dual description for cogroup objects in terms of **corepresentable functors**, those of the form $\mathrm{Hom}_{\mathcal{C}}(X, -)$.

Definition 3.4. A cogroup object in a locally small category $\mathcal{C}$ is an object X such that the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ has a group structure for each object Y. In other words, there is a lift of the corepresentable functor:

$$\begin{tikzcd} & \underline{\mathrm{Grp}} \arrow[d, "U"] \\ \mathcal{C} \arrow[ur, dashed, "\mathrm{Hom}_{\mathcal{C}}(X,-)"] \arrow[r] & \underline{\mathrm{Set}} \end{tikzcd}$$

Perhaps an interesting thing to note is that endowing an object X with a cogroup structure is *not* the same as lifting the corepresentable functor $\mathrm{Hom}_{\mathcal{C}}(X,-)$ from $\underline{\mathrm{Set}}$ to $\underline{\mathrm{CoGrp}}$. Focusing on the comultiplication $\Delta : X \to X \coprod X$, we get that the map induced on Hom is contravariant, given by precomposition. Thus, the natural transformation associated to Δ is

$$\mathrm{Hom}_{\mathcal{C}}(X \coprod X, -) = \mathrm{Hom}_{\mathcal{C}}(X,-) \times \mathrm{Hom}_{\mathcal{C}}(X,-) \to \mathrm{Hom}_{\mathcal{C}}(X,-).$$

These reworked definitions will be useful for generalization, but are also useful in their own right. For example, suppose X is an affine scheme over k, so that $X = \mathrm{Spec}(A)$ for some $A \in \underline{k\text{-Alg}}$. Since $\underline{\mathrm{Aff}_k}$ is locally small, we can represent X by its representable functor, which we will call the **functor of points**,

$$\mathrm{Hom}_{\underline{\mathrm{Aff}_k}}(-,X) : \underline{\mathrm{Aff}_k} \to \underline{\mathrm{Set}}.$$

An **affine group scheme** is a group object $G \in \underline{\mathrm{Aff}_k}$. We may then interpret this as the data of maps of schemes $m : G \times_k G \to G$, $e : \mathrm{Spec}(0) \to G$, and $i : G \to G$, or as a lift of the functor of points to groups. But in this context we can go further: there is an equivalence of categories $(\underline{\mathrm{Aff}_k})^{op} \cong \underline{k\text{-Alg}}$. Thus every affine scheme is uniquely determines and is determined by its coordinate algebra, and so we could equivalently study the functor

$$\mathrm{Hom}_{\underline{k\text{-Alg}}}(A,-) : \underline{k\text{-Alg}} \to \underline{\mathrm{Set}}.$$

If X is an affine group scheme, then contravariance tells us that its coordinate algebra A is a cogroup algebra in $\underline{k\text{-Alg}}$; in other words, the coordinate algebra for an affine group scheme is a Hopf algebra! More ways to say this is that an affine group scheme is the spectrum of a Hopf algebra, and that an Hopf algebra is a representable presheaf of groups on $\underline{k\text{-Alg}}$.

3.2. **Groupoids and cogroupoids.** We saw in the previous section that a Hopf algebra is a cogroup object in $\underline{k\text{-Alg}}$. We will use this as an analogy to define Hopf algebroids. First, we make a simple observation. If G is a group, then we may regard G as a one object category, denoted $\underline{G}$, where the set of morphisms $\mathrm{Hom}_G(*,*)$ is given by the elements of G and composition is given by multiplication. Notice that every morphism in this category is an isomorphism precisely because every element in G has a two-sided inverse. In fact, this provides an example of a groupoid!

Definition 3.5. A **groupoid** is a small category where every morphism is invertible.

We want to generalize this notion a little. What is the structure of such a category $\mathcal{C}$? In the case where we abstracted a group, we saw that every morphism had the same source and target, but this need not be true. So, we ought to have some sort of "source and target" morphisms, defined on Hom sets. Furthermore, we had a composition law given by the multiplication on the group. Again, this was nontrivial in the case of having one object, but is not in general, and so we should have some composition law telling us that if two morphisms had agreeing target and source, then they are able to be composed. We also had an identity morphism given by the identity element of G, and so we should have an identity element for each object in $\mathcal{C}$. Finally, each element is invertible, so every morphisms between any two objects should have an inverse. We then make a restated definition.

Definition 3.6. A **groupoid object** in a category $\mathcal{C}$ is the data of two objects X_0 (the objects) and X_1 (the morphisms) together with structure maps

$$s,t : X_1 \to X_0, \quad e : X_0 \to X_1, \quad m : X_1 \times_{X_0} X_1 \to X_1, \quad i : X_1 \to X_1,$$

where $X_1 \times_{X_0} X_1$ is the pullback

$$\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "t"] \\
X_1 \arrow[r, "s"] & X_0
\end{tikzcd}$$

such that the expected diagrams commute:

$$\begin{tikzcd}
& X_1 \arrow[dr, "s"] & \\
X_0 \arrow[ur, "e"] \arrow[rr, "\mathrm{id}"] \arrow[dr, "e"'] & & X_0 \\
& X_1 \arrow[ur, "t"'] &
\end{tikzcd} , \quad
\begin{tikzcd}
& X_1 \arrow[dr, "s"] & \\
X_1 \times_{X_0} X_1 \arrow[ur, "m"] \arrow[dr, "p_1"'] & & X_0 \\
& X_1 \arrow[ur, "s"'] &
\end{tikzcd} , \quad
\begin{tikzcd}
& X_1 \arrow[dr, "t"] & \\
X_1 \times_{X_0} X_1 \arrow[ur, "m"] \arrow[dr, "p_2"'] & & X_0 \\
& X_1 \arrow[ur, "t"'] &
\end{tikzcd} ,$$

where p_1 and p_2 are projections,

$$\begin{tikzcd}
X_1 \times_{X_0} X_1 \times_{X_0} X_1 \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"'] & X_1 \times_{X_0} X_1 \arrow[d, "m"] \\
X_1 \times_{X_0} X_1 \arrow[r, "m"] & X_1
\end{tikzcd} , \quad
\begin{tikzcd}
& X_1 \times_{X_0} X_1 \arrow[dr, "m"] & \\
X_1 \arrow[ur, "(\mathrm{id}, e \circ t)"] \arrow[rr, "\mathrm{id}"] \arrow[dr, "(e \circ s, \mathrm{id})"'] & & X_1 \\
& X_1 \times_{X_0} X_1 \arrow[ur, "m"'] &
\end{tikzcd} ,$$

and finally the unit axioms,

$$\begin{tikzcd}
X_1 \arrow[r, "i"] \arrow[dr, "\mathrm{id}"'] & X_1 \arrow[d, "i"] \\
& X_1
\end{tikzcd} , \quad
\begin{tikzcd}
X_1 \arrow[r, "t"] \arrow[dr, "s"'] & X_1 \arrow[d, "i"] & X_1 \arrow[l, "s"'] \arrow[dl, "t"] \\
& X_1 &
\end{tikzcd} , \quad
\begin{tikzcd}
X_1 \arrow[r, "(\mathrm{id}, i)"] \arrow[d, "s"'] & X_1 \times_{X_0} X_1 \arrow[d, "m"] & X_1 \arrow[l, "(i, \mathrm{id})"'] \arrow[d, "t"] \\
X_0 \arrow[r, "e"] & X_1 & X_0 \arrow[l, "e"']
\end{tikzcd} .$$

A nice and suggestive way to represent this data is as the following diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "e" description] & X_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] \arrow[loop above, "i"] \arrow[r, shift left=2] \arrow[r, shift right=2] & X_1 \times_{X_0} X_1 \arrow[l, "m" description]
\end{tikzcd}$$

The two unlabeled maps $X_1 \to X_1 \times_{X_0} X_1$ include against the identity morphism, taking the form $f \mapsto (f, \mathrm{id})$ and $f \mapsto (\mathrm{id}, f)$. Note that there are also two maps coming in the opposite direction $X_1 \times_{X_0} X_1 \to X_1$ which project onto the first or second factor: $(f, g) \mapsto f$ and $(f, g) \mapsto g$. Additionally, we can continue to extend this diagram out to the right by forming the iterated fiber product $X_1 \times_{X_0} \cdots \times_{X_0} X_1$. All the structure maps will generalize the structure maps between X_1 and $X_1 \times_{X_0} X_1$: we have face maps which include against the identity, and we have degeneracy maps which either compose pairwise morphisms or project opposite a morphism in the first or last component. Notice that this simplicial object is completely determined by its degree 0 and 1 components.

This presentation highlights an important fact. The construction we were performing was the **nerve** construction, which can be performed for any category in the exact same way, and produces a simplicial set from the objects and morphisms of $\mathcal{C}$.

Proposition 3.1. *The nerve construction produces a functor*

$$N : \underline{Grpd} \to \underline{sSet},$$

exhibiting $\underline{Grpd}$ as a full subcategory of $\underline{sSet}$. Moreover, the nerve of a groupoid is a Kan complex.

Groupoid objects are not uncommon. The prototypical example is the **fundamental groupoid**. If X is a space, then we could certainly turn the fundamental group $\pi_1(X)$ into a groupoid as we can do for any group, but there is another construction which is more interesting. We can let $\Pi_1(X)$ be the groupoid whose objects are the points of X and whose morphisms are homotopy classes of maps. The structure maps are clear. We can certainly pick a source and target of any morphism and choose a constant map as our identity $e : \Pi_1(X)_0 \to \Pi_1(X)_1$. We can also compose maps, which is well defined as the homotopy class of a composition $f \circ g$ only depends on the homotopy classes of f and g. Finally, for any path $f : p \mapsto q$, we can form the path which traverses backwards, giving $f^{-1} : q \mapsto p$. Notice that $\Pi_1(X)(p,p) = \pi_1(X,p)$ for any point $p \in X$. We can form another example using homological algebra. Take a chain complex with two terms $C_1 \to C_0$. We can form a groupoid where the objects are given by the term C_0, and the morphisms are given by the set $C_1 \oplus C_0$. The source map $s : C_1 \oplus C_0 \to C_0$ is projection onto the C_0 coordinate, and the target map $t : C_1 \oplus C_0 \to C_0$ is given by $t(c_1 + c_0) = d(c_1) + c_0$. The identity map is given by $e(c_0) = c_0$, and multiplication and inverse are defined componentwise.

Now, we would like to reverse all the arrows. In fact, this will lead us in the direction we need to head.

Definition 3.7. A **cogroupoid object** in a category $\mathcal{C}$ is the data of two objects Y_0, Y_1 together with structure maps

$$\eta_L, \eta_R : Y_0 \to Y_1, \quad \varepsilon : Y_1 \to Y_0, \quad \Delta : Y_1 \to Y_1 \coprod_{Y_0} Y_1 \quad c : Y_1 \to Y_1$$

called the left and right units, counit, comultiplication, and conjugation or coinverse, such that expected diagrams (dual to those for a groupoid) commute.

As this is formally dual to a groupoid, it is not surprising that the data of a cogroupoid can be represented in a similar diagram:

$$\begin{tikzcd} Y_0 \arrow[r, shift left=2, "\eta_L"] \arrow[r, shift right=2, "\eta_R"'] & Y_1 \arrow[l, "\varepsilon"] \arrow[loop above, "c"] \arrow[r, "\Delta"] & Y_1 \coprod_{Y_0} Y_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

Similar to before, we can extend this diagram infinitely out to the right to create a cosimplicial object which is completely determined by its degree 0 and 1 components. This is completely analogous to the situation we have with E_1-page of the Adams spectral sequence when E is a flat ring spectrum, as we saw in section 2. We will hone in on a particular class of cogroupoids.

Definition 3.8. A **Hopf algebroid** is a cogroupoid object in the category $\underline{k\text{-Alg}}$.

This is completely analogous to a Hopf algebra being a cogroup object in $\underline{k\text{-Alg}}$. We will describe this object in many ways in the next section.

Remark 3.9. One could also define a Hopf algebra/algebroid as a group/groupoid object in $\underline{k\text{-Coalg}}$.

3.3. **Hopf algebroids.** We saw in the last section that a Hopf algebroid is a cogroupoid object in $\underline{k\text{-Alg}}$. Let us unpack that data. This consists of two k-algebras A and Γ, which we will refer to as (A, Γ). There are two unit maps $\eta_L, \eta_R : A \to \Gamma$, a counit $\varepsilon : \Gamma \to A$, a comultiplication $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$, and a conjugation $c : \Gamma \to \Gamma$. Again, this data fits into a cosimplicial diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "\eta_L"] \arrow[r, shift right=2, "\eta_R"'] & \Gamma \arrow[l, "\varepsilon"] \arrow[loop above, "c"] \arrow[r, "\Delta"] & \Gamma \otimes_A \Gamma \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

While not technically required for the definition, it is often the case that the left unit is flat. **We assume now that the left unit is flat.**

Just as we saw with cogroup objects, it is often nicer and more enlightening to phrase things in terms of corepresentable functors. The pair of k-algebras (A, Γ) being a Hopf algebroid is the same data as a natural groupoid structure on $(\mathrm{Hom}_{\underline{k\text{-Alg}}}(A, R), \mathrm{Hom}_{\underline{k\text{-Alg}}}(\Gamma, R))$ for every k-algebra R. In other words,

Definition 3.10. A Hopf algebroid is a pair of k-algebras (A, Γ) which form a functor

$$(\mathrm{Hom}_{\underline{k\text{-Alg}}}(A, -), \mathrm{Hom}_{\underline{k\text{-Alg}}}(\Gamma, -)) : \underline{k\text{-Alg}} \to \underline{\mathrm{Grpd}}$$

lifting the pair of corepresentable functors landing in $\underline{\mathrm{Set}}$.

Just as Hopf algebras are the coordinate algebras of affine group schemes, so too are Hopf algebroids the algebraic data of an **affine groupoid scheme**. An affine groupoid scheme is a groupoid object $(X_0, X_1) \in \underline{\mathrm{Aff}_k}$. To any Hopf algebroid (A, Γ), the pair $(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$ is the corresponding affine groupoid scheme. To be pedantic, the data of the an affine groupoid scheme can be placed into a simplicial diagram:

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r, "e" description] & \mathrm{Spec}(\Gamma) \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] \arrow["i", loop above] \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathrm{Spec}(\Gamma) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\Gamma) \arrow[l, "m" description] \end{tikzcd}$$

We could equivalently say that a Hopf algebroid is a presheaf of groupoids on the category $\underline{k\text{-Alg}}$. As a note, we see that if the left and right unit for a Hopf algebroid agree, then in fact (A, Γ) for an honest Hopf algebra. Thinking of a Hopf algebroid as a groupoid scheme, this is the same as saying that the source and target of each "morphism" in $\mathrm{Spec}(\Gamma)$ is the same, or that (A, Γ) is the originaly example of a groupoid as a one object category obtained from a group. Thus we may identify (A, Γ) with an honest group object in affine schemes, i.e. a group scheme, or equivalently a Hopf algebra.

Remark 3.11. What happens if we sheafify a Hopf algebroid, viewed as a presheaf of groupoids on $\underline{k\text{-Alg}}$? This is a **stack**, which we will not discuss in these notes.

The point of all the previous abstractions is threefold.

(1) We want to better understand the E-Adams spectral sequence for E a flat ring spectrum.
(2) The functor of points gives useful tools to understand (co)group(oids), and gives us a dictionary to pass from k-algebras to affine k-schemes.
(3) We have already seen an example of a Hopf algebroid.

The last point should come as no surprise if one compares the E_1-page of the E-Adams spectral sequence and the cosimplicial diagram related to a Hopf algebroid.

Theorem 3.1. *Let E be a flat ring spectrum. Then the pair (E_*, E_*E) is a Hopf algebroid.*

We have already discussed the left and right unit maps $\eta_L, \eta_R : E_* \to E_*E$, and we saw that the assumption of flatness is precisely what allows us to rewrite the E_1-page of the E-Adams spectral sequence as the cosimplicial diagram associated to a Hopf algebroid. The only thing we have not explicitly discussed is the conjugation $c : E_*E \to E_*E$; this just comes from applying π_* to the swap map $E \otimes E \to E \otimes E$.

Remark 3.12. The definition of a Hopf algebroid does *not* require that the left unit map $\eta_L : A \to \Gamma$ be a flat map of rings. However, if E is a ring spectrum, then (E_*, E_*E) only forms a Hopf algebroid when $\eta_L : E_* \to E_*E$ is flat. Since these are the examples we care about, most of what we will say about Hopf algebroids will only apply in these circumstances.

To fully identify the E_1-page, we need to discuss some aspects of homological algebra over a Hopf algebroid.

Definition 3.13. A **comodule** over a Hopf algebroid (A, Γ) is an A-module M with an A-module map called coaction:

$$\psi : N \to \Gamma \otimes_A N.$$

This coaction is counital and coassociative, given by the following commutative diagrams:

$$\begin{tikzcd} N \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}"'] & \Gamma \otimes_A N \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\ & N \end{tikzcd}, \qquad \begin{tikzcd} N \arrow[r, "\psi"] \arrow[d, "\psi"'] & \Gamma \otimes_A N \arrow[d, "\Delta\otimes\mathrm{id}"] \\ \Gamma \otimes_A N \arrow[r, "\mathrm{id}\otimes\psi"] & \Gamma \otimes_A \Gamma \otimes_A N \end{tikzcd}.$$

One thing to note here is that we can define a comodule over any coalgebraic structure in a similar fashion. If we take H to be an honest Hopf algebra, for example, then there is a category $\underline{H\text{-Comod}}$. Using the functor of points, one can show that an H-comodule structure on some k-vector space V is the exact same data as giving V the structure of a $\mathbb{G} = \operatorname{Spec}(H)$-representation, using that $\operatorname{Spec}(H)$ has the structure of an affine group scheme. This means that for each $R \in \underline{k\text{-Alg}}$ there is a linear map

$$\mathbb{G}(R) \times (V \otimes R) \to (V \otimes R).$$

If we look at the Hopf algebroid (E_*, E_*E) associated to some flat ring spectrum E, then E_*X is a comodule over (E_*, E_*E), and the maps on the E_1-page of the E-Adams spectral sequence come from the coaction. Recall that this map

$$\psi : E_*X \to E_*E \otimes_{E_*} E_*X$$

is the map obtained by applying π_* to

$$E \otimes X \cong E \otimes \mathbb{S} \otimes X \xrightarrow{\mathrm{id}\otimes e\otimes\mathrm{id}} E \otimes E \otimes X.$$

The category $\underline{\Gamma\text{-Comod}}$ is the category of all (A, Γ)-comodules. We cite the the next Theorem with no proof.

Theorem 3.2. *If (A, Γ) is a Hopf algebroid such that $\eta_L : A \to \Gamma$ is flat, then $\underline{\Gamma\text{-Comod}}$ is an abelian category with enough injectives.*

This theorem tells us that it makes sense to talk about derived functors over a Hopf algebroid. Moreover, it tells us that the homological algebra we developed in spectra, the E-Adams resolution, gives us data to work with in $\underline{E_*E\text{-Comod}}$.

Definition 3.14. Let (A, Γ) be a Hopf algebroid and N a Γ-comodule. The **cobar complex** $C^\bullet_\Gamma(N)$ is a cochain complex taking the form

$$0 \to N \xrightarrow{\psi} \overline{\Gamma} \otimes_A N \xrightarrow{\Delta-\psi} \overline{\Gamma} \otimes_A \overline{\Gamma} \otimes_A N \to \cdots,$$

where $\overline{\Gamma} = \ker(\varepsilon : \Gamma \to A)$ is the kernel of the augmentation ideal.

Proposition 3.2. *The cobar complex computes cohomology. That is,*

$$H^*(C^\bullet_\Gamma(N)) = \operatorname{Ext}_\Gamma(A, N).$$

Proof. There is a resolution of N by cofree (A, Γ)-comodules given by

$$0 \to N \to \Gamma \otimes_A \overline{\Gamma} \otimes_A N \to \Gamma \otimes_A \overline{\Gamma} \otimes_A \overline{\Gamma} \otimes N \to \ldots$$

with differential given by the coaction on N, the counit $A \to \Gamma$ and the coproduct on $\overline{\Gamma}$. We can thus compute $\operatorname{Ext}_\Gamma(A, N)$ as the cohomology of the complex $\operatorname{Hom}_{\underline{\Gamma\text{-Comod}}}(A, \Gamma \otimes_A \overline{\Gamma}^{\otimes_A \bullet} \otimes_A N)$. By base change, or better yet by the cofree-forgetful adjunction between $\underline{\Gamma\text{-Comod}}$ and $\underline{A\text{-mod}}$, this is the same as the cohomology of $\operatorname{Hom}_{\underline{A\text{-mod}}}(A, \overline{\Gamma}^{\otimes_A \bullet} \otimes_A N)$, which is isomorphic to the cobar complex $C^\bullet_\Gamma(N)$. $\square$

4. Adams spectral sequence II

We're now in a good place to interpret the Adams spectral sequence in many different ways.

4.1. **The E_2-page.** Let E be a flat ring spectrum. We have seen that we can construct an E-Adams spectral sequence for any spectrum X. This is a spectral sequence computing information related to $\pi_* X$, and we saw that its E_1-page takes the form

$$E_*X \to E_*E \otimes_{E_*} E_*X \to E_*E \otimes_{E_*} \otimes E_*E \otimes_{E_*} E_*X \to \cdots$$

We have also identified the object (E_*, E_*E) as a Hopf algebroid, and the homology E_*X as a comodule over this Hopf algebroid. In fact, it is not hard to see that the E_1-page is actually quasi-isomorphic to the cobar complex $C^\bullet_{E_*E}(E_*X)$. The cohomology of the cobar complex then gives the E_2-page of the E-Adams spectral sequence.

Theorem 4.1. *Let E be a flat ring spectrum, X any spectrum. Then the E_2-page of the E-Adams spectral sequence takes the form*

$$E_2 = \mathrm{Ext}_{E_*E}(E_*, E_*X),$$

where this Ext *is taken in the category $\underline{E_*E\text{-Comod}}$.*

We also have duality: to the Hopf algebroid (E_*, E_*E), we have an associated affine groupoid scheme over $\mathrm{Spec}(E_*)$, which we can denote by $\mathbb{G}_E$. Just as with affine groups, the (E_*, E_*E)-comodule structure on E_*X uniquely gives E_*X the structure of a representation of the affine groupoid scheme $\mathbb{G}_E$. This lets us rephrase the previous theorem:

Theorem 4.2. *With the same assumptions as above, we may also identify the E_2-page of the E-Adams spectral sequence as the cohomology of the groupoid $\mathbb{G}_E$ with coefficients in the representation E_*X:*

$$E_2 = H^*(\mathbb{G}_E, E_*X).$$

If X itself is a ring spectrum, then $\mathbb{G}_E$-action coming from the (E_*, E_*E)-comodule structure on E_*X translates to an action of schemes:

$$\mathbb{G}_E \times_{\mathrm{Spec}(E_*)} \mathrm{Spec}(E_*X) \to \mathrm{Spec}(E_*X).$$

In this case, we can further identify the E_2-page as the cohomology of the GIT-quotient (i.e. the orbit space) $\mathrm{Spec}(E_*X)//\mathbb{G}_E$ with coefficients in the structure sheaf $\mathcal{O}_{\mathrm{Spec}(E_*X)//\mathbb{G}_E}$. Note that when E_*E has the structure of an honest Hopf algebra, this argument completely holds using the theory of affine group schemes (and really makes more sense).

Remark 4.1. I am highlighting the connection to affine schemes here because of the relationship that Hopf algebroids have with stacks. One can suitably understand the E_2-page as the cohomology of the stack arising from a Hopf algebroid with coefficients in the quasi-coherent sheaf arising from E_*X, and it becomes very useful very quickly to use the language of schemes. Much of modern homotopy is better viewed in this light.

Remark 4.2. One should be more careful when setting up these spectral sequences in terms of affine schemes than I am. To a complex-oriented spectrum E, its formal group law is associated to a *formal* group scheme, and so this cohomology may be better interpreted as the cohomology the formal group scheme associated to E.

4.2. **Example: the classical Adams spectral sequence.** We will highlight one example. There is a deeper connection with stacks than I will present here, but one can at least present the E_2-page in terms of a Hopf algebroid without going to deep into the theory of formal group laws (something excluded from this talk). For E a general complex-oriented spectrum, we will need this connection with formal group laws to make the most concise statement.

Let $E = \mathrm{H}\mathbb{F}_2$ be the mod-2 Eilenberg-Maclane spectrum. This is a ring spectrum, since $\mathbb{F}_2$ is a ring, and we can look at its Adams spectral sequence. Let's do this for the simplest case, when $X = \mathbb{S}$. The E_1-page takes the form

$$\mathbb{F}_2 \to (H\mathbb{F}_2)_*(H\mathbb{F}_2) \to (H\mathbb{F}_2)_*(H\mathbb{F}_2 \otimes H\mathbb{F}_2) \to \cdots$$

If we want to pass to the E_2-page in a nice way, we should determine if the left unit map $\eta_L : \mathbb{F}_2 \to (H\mathbb{F}_2)_*(H\mathbb{F}_2)$ is flat. This is certainly true, though, since $\mathbb{F}_2$ is a field, and any graded module is free, hence flat. This map can also be shown to be flat explicitly. We know the algebra $(H\mathbb{F}_2)_*(H\mathbb{F}_2) := A^\vee$ as the **dual Steenrod algebra**, and it is a result of Milnor that, as a Hopf algebra,

$$A^\vee = \mathbb{F}_2[\xi_1, \xi_2, \ldots], \quad \Delta(\xi_k) = \sum_{i+j=k} \xi_i \otimes \xi_j^{2^i}.$$

This is a commutative and non-cocommutative Hopf algebra. One way to see that we get a Hopf algebra instead of a Hopf algebroid comes from the same reason that we get flatness: the left unit map is nonzero, and hence must send 1 to 1. Thus the action of $\mathbb{F}_2$ on $A^\vee$ is the usual one, so $A^\vee$ is flat over $\mathbb{F}_2$. Moreover, it is forced upon us that $\eta_L = \eta_R$, so that $A^\vee$ is an honest Hopf algebra. The augmentaion is defined by $\varepsilon(\xi_i) = 0, \varepsilon(1) = 1$, and the conjugation is defined recursively, letting $\xi_0 := 1$:

$$c(\xi_0) = 1, \sum_{0 \le i \le n} \xi_{n-i}^{2^i} c(\xi_i) = 0.$$

For example, $c(\xi_1)$ has the property that $\xi_1 + c(\xi_1) = 0$, hence $c(\xi_1) = \xi_1$, and $c(\xi_2)$ has the property that

$$\xi_2 + \xi_1^2 + c(\xi_2) = 0,$$

so that $c(\xi_2) = \xi_2 + \xi_1^2$.

We can pass to the E_2-term of the Adams spectral sequence:

$$E_2 = \mathrm{Ext}_{A^\vee}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*\mathbb{S}.$$

There are many methods for computing this E_2-term. One can analyze the cobar complex $C^\bullet_{A^\vee}(\mathbb{F}_2)$, but this quickly gets both tedious and cumbersome. May introduced the eponymous May spectral sequence by filtering the cobar complex by powers of the augmentation co-ideal, and this algebraic spectral sequence converges to the E_2-page of the Adams spectral sequence. There are also change-of-rings isomorphisms which allow one to compute portions of the E_2-page by comparing against Adams spectral sequences for particular spectra X whose mod-2 homology witness a certain sub-Hopf algebra of $A^\vee$. Instead of focusing on computing this E_2-page (which is very fun but very hard and tangential to this talk), we will give another interpretation of the E_2-page.

The algebra $A^\vee$ is a Hopf algebra, and so naturally defines an affine group scheme $\mathbb{G} := \mathrm{Spec}(A^\vee)$ over $\mathbb{F}_2$. For any spectrum X, the mod-2 homology H_*X is a comodule over $A^\vee$, so naturally gives a representation of $\mathbb{G}$. Thus, we may identify the E_2-page of the Adams spectral sequence as the group cohomology

$$E_2 = H^*(\mathbb{G}, H_*X).$$

In fact, using formal groups and formal group laws, one can show that this affine group scheme as the group scheme of degree-preserving isomorphisms of the additive formal group $\mathbb{G}_a$.

Remark 4.3. A similar analysis can be done to understand the $H\mathbb{F}_p$-Adams spectral sequence, although the odd primary dual Steenrod algebra is not polynomial. For any p, the mod-p Adams spectral sequence converges to $\pi_*\mathbb{S}_p^\wedge$. It is arguably through this spectral sequence that we know the most information about the stable homotopy groups of spheres.

Remark 4.4. The MU or BP-Adams spectral sequence is called the **Adams-Novikov spectral sequence**. When dealing with complex-oriented spectra and chromatic homotopy theory, these spectral sequences play a large computational and conceptual role.

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