Cooperations in motivic homotopy theory

Jackson Morris

Thesis Defense

May 21, 2026

homotopy theory?

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“The study of spaces up to continuous deformation”

homotopy theory?

Think: what types of "holes" does my space have?

"The study of spaces up to continuous deformation"

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To a homotopy theorist,
a coffee cup is the same as a donut!

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A way to "measure holes":
homotopy groups
$$\pi_n(X) = [S^n, X]$$

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the spheres $S^n \subseteq \mathbb{R}^{n+1}$

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Question:

What are the homotopy groups of spheres?

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A way to "measure holes": homotopy groups

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Question:

What are the homotopy groups of spheres?

Can we understand $\pi_n(S^k)$?

Sometimes!

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$\pi_i(S^n)$	S^1	S^2	S^3	S^4	S^5	S^6	S^7
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$\pi_3(-)$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
$\pi_4(-)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0
$\pi_5(-)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0
$\pi_6(-)$	0	$\mathbb{Z}/12$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
$\pi_7(-)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$
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Define the k^{th} stable stem

$$\pi_k(\mathbb{S}) = \mathrm{colim}_n \pi_{n+k}(S^n)$$

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The **A**dams **S**pectral **S**equence

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$$\begin{aligned}
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A tool which turns homological algebra into homotopy groups

The

Adams

Spectral

Sequence

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huge algebra with infinitely many generators

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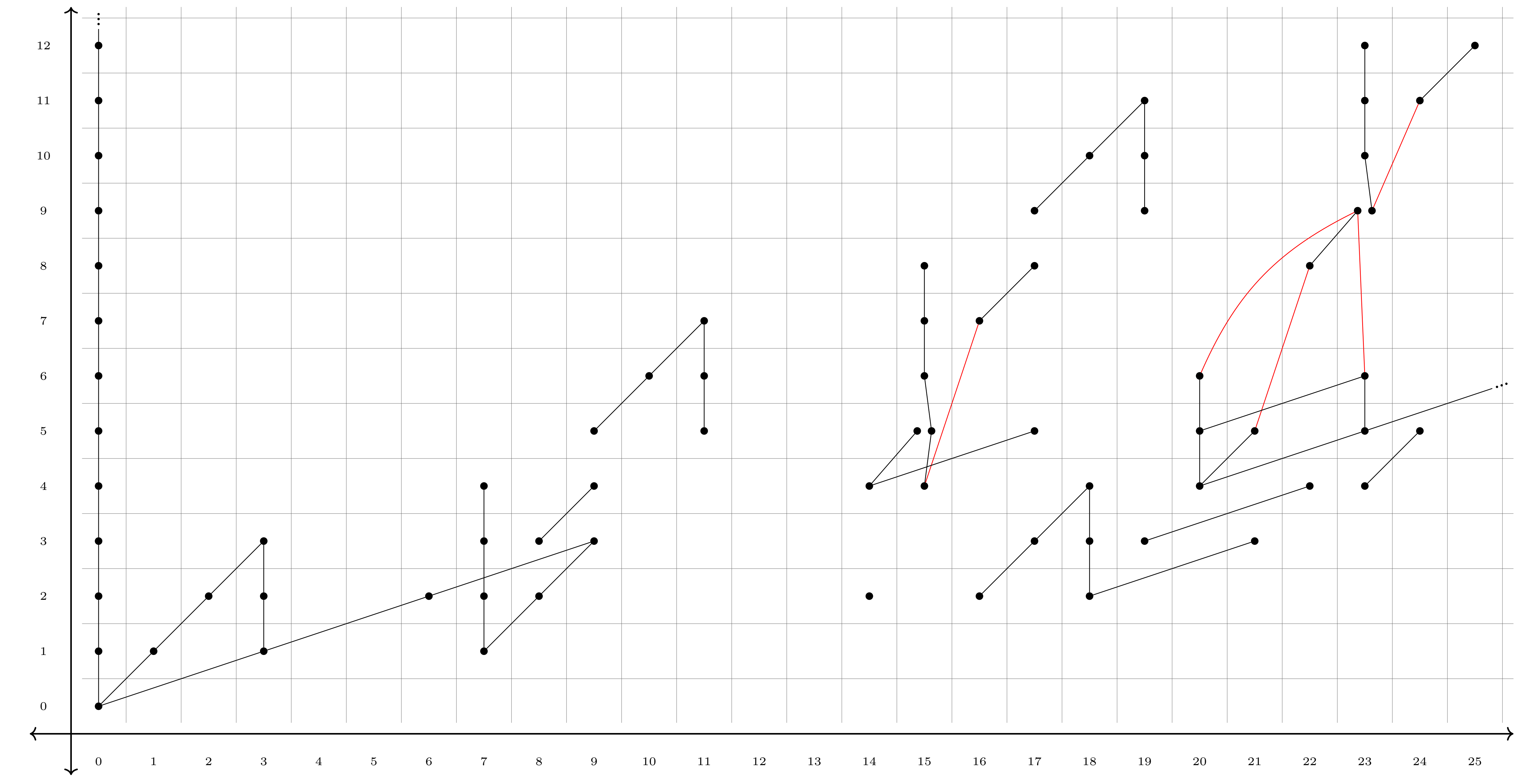

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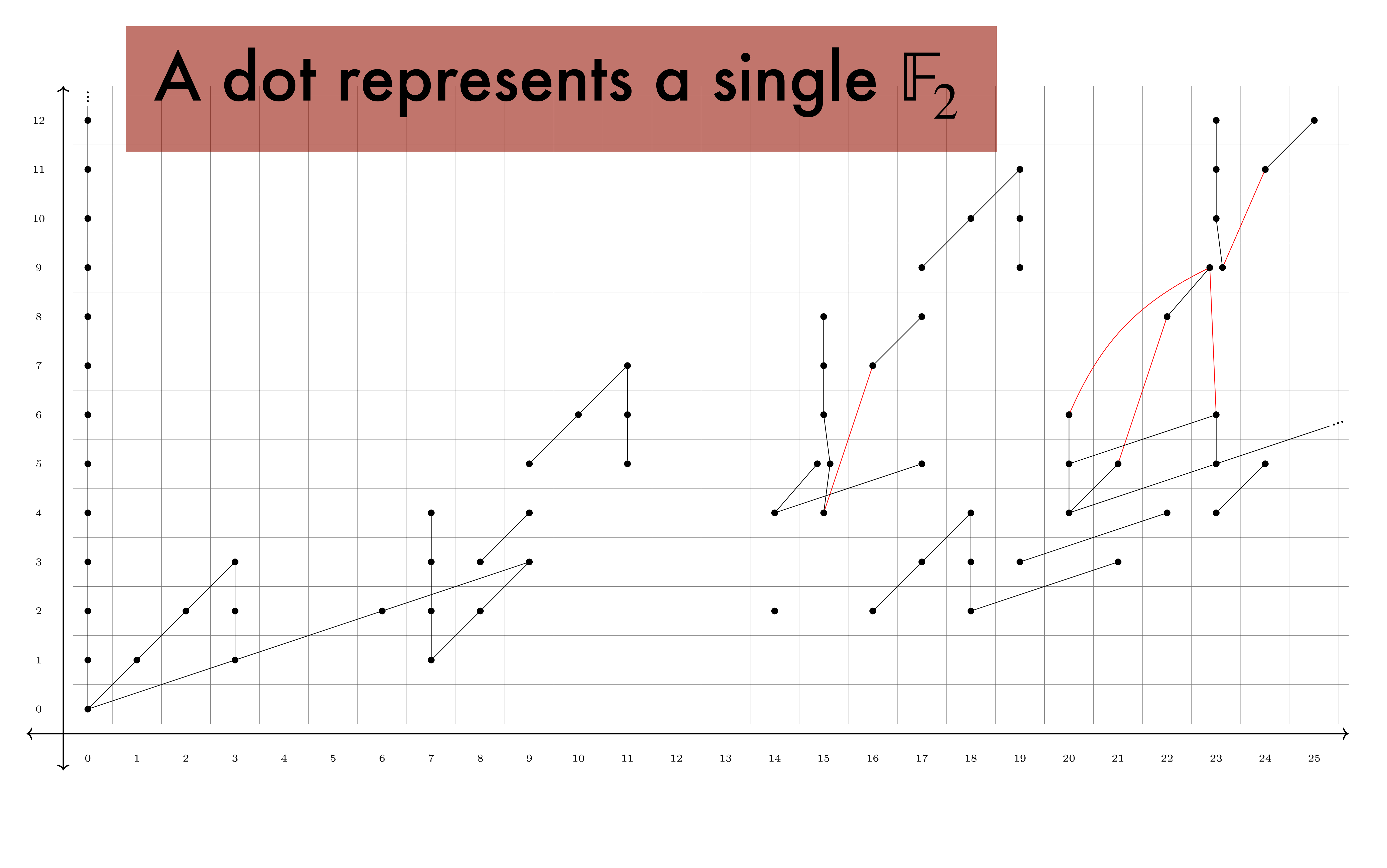

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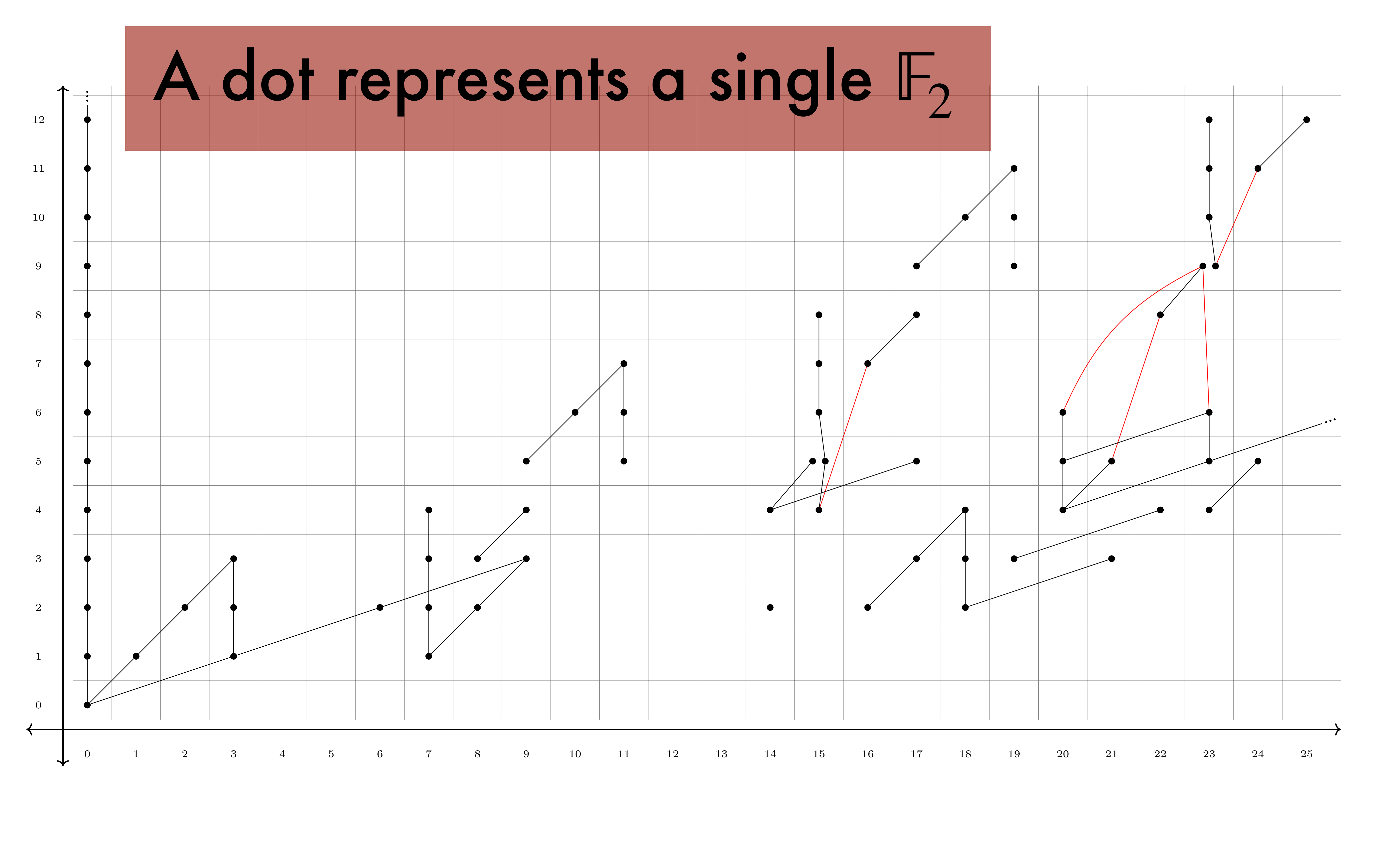

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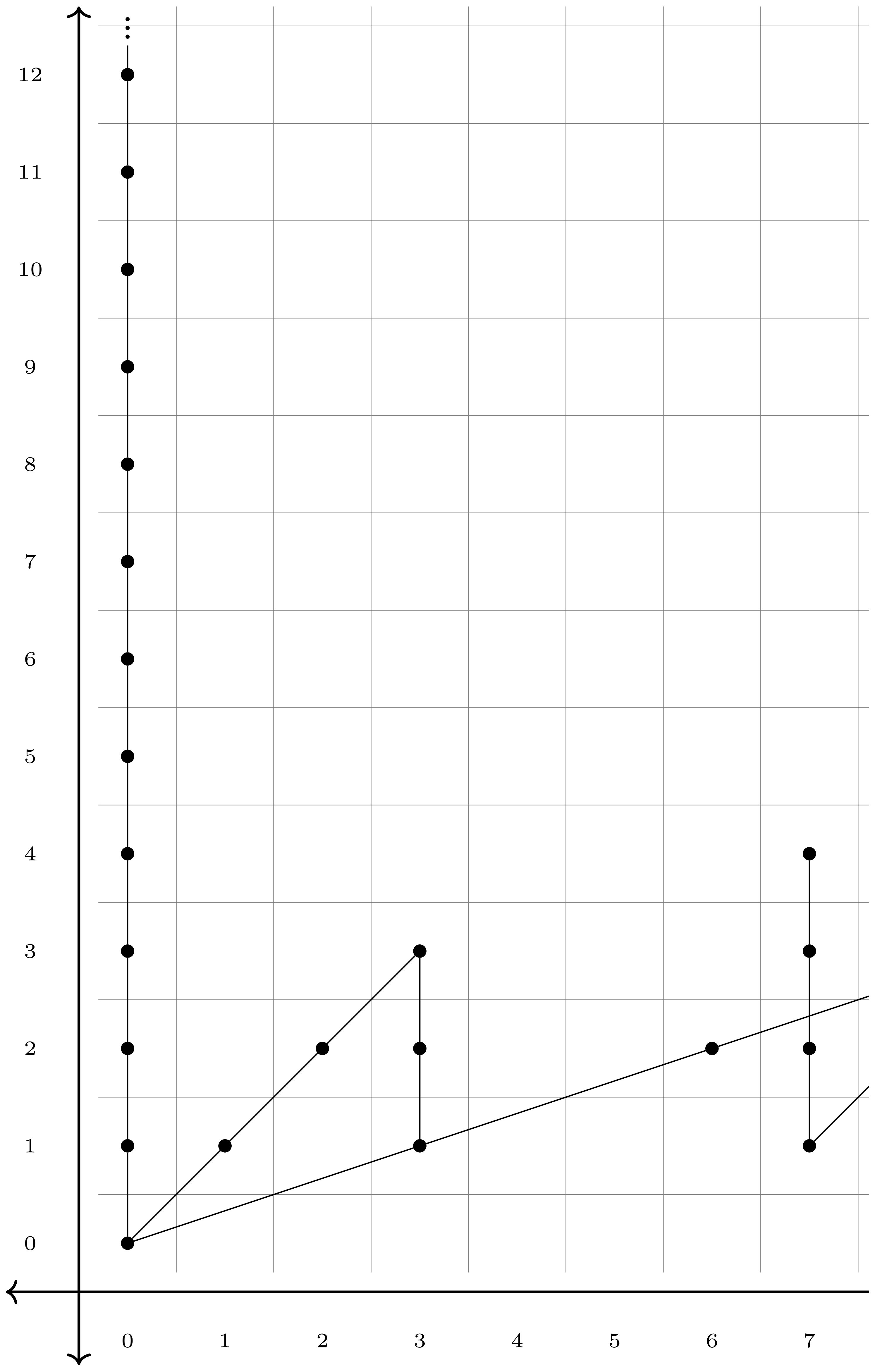
Columns give stable stems!
(Working at prime 2 now)



12
11
10
9
8
7
6
5
4
3
2
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0
0
1
2
3
4
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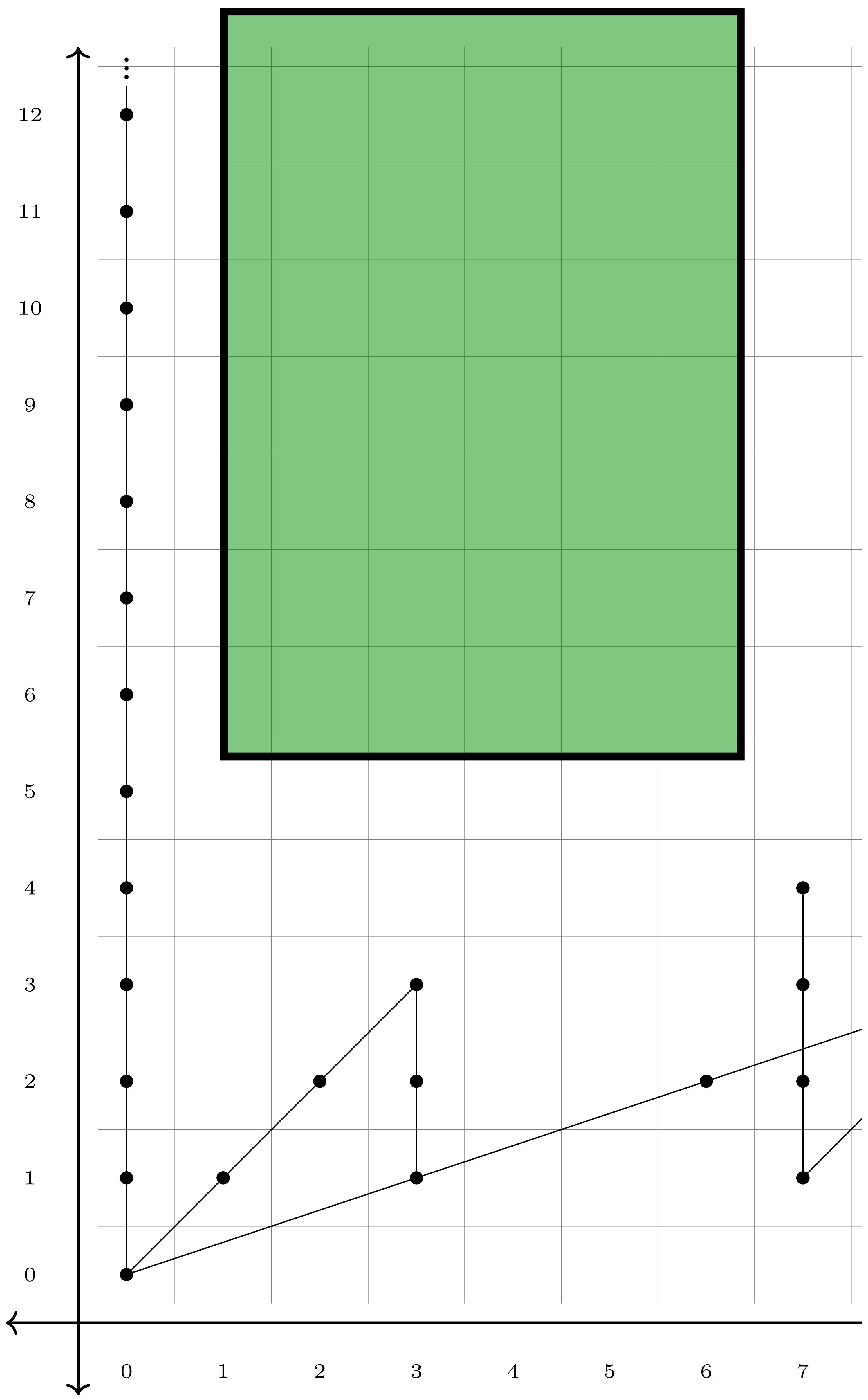
6
7



12
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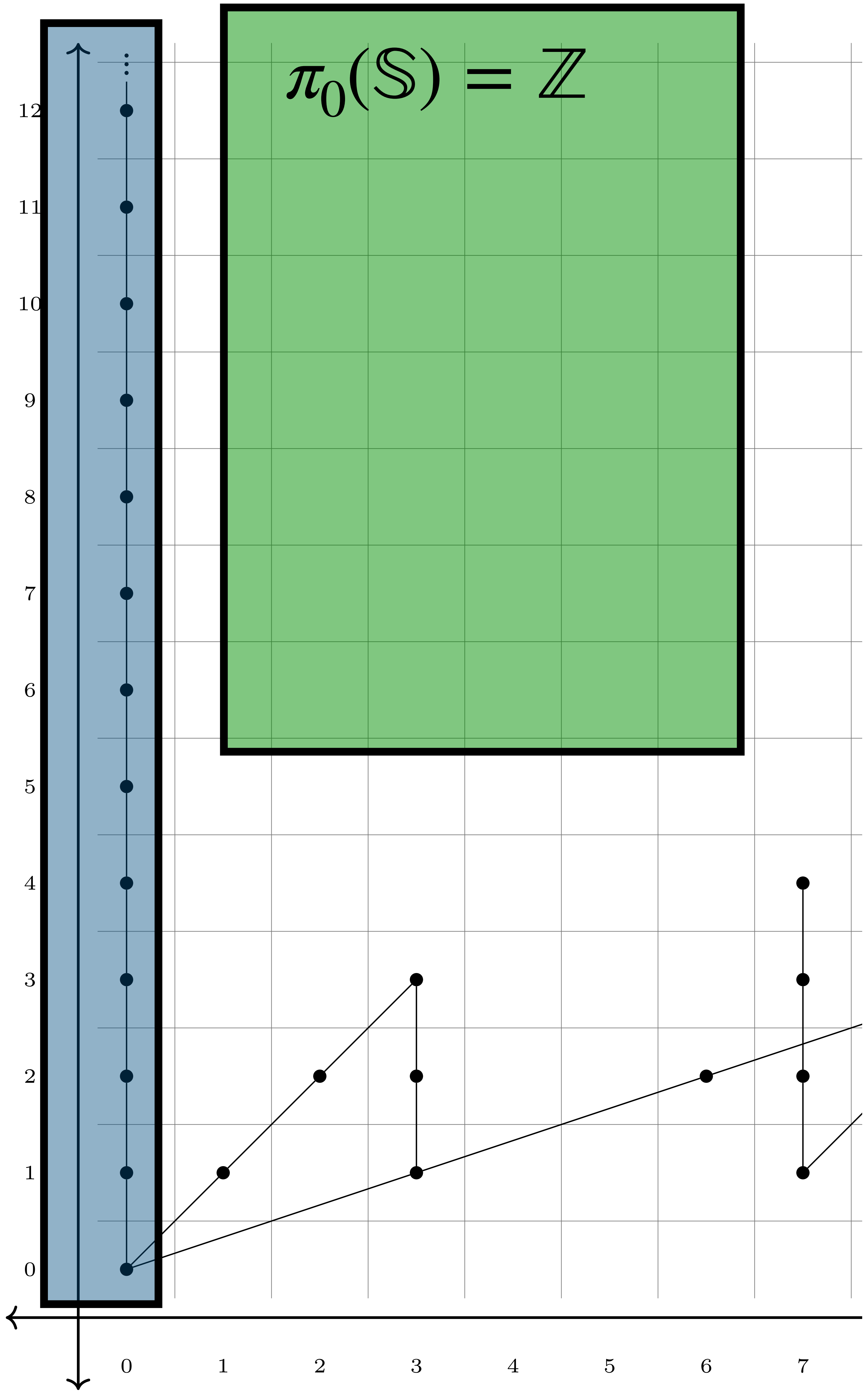
0 1 2 3 4 5 6 7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0
1
2
3
4
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6
7
8
9
10
11
12
0
1
2
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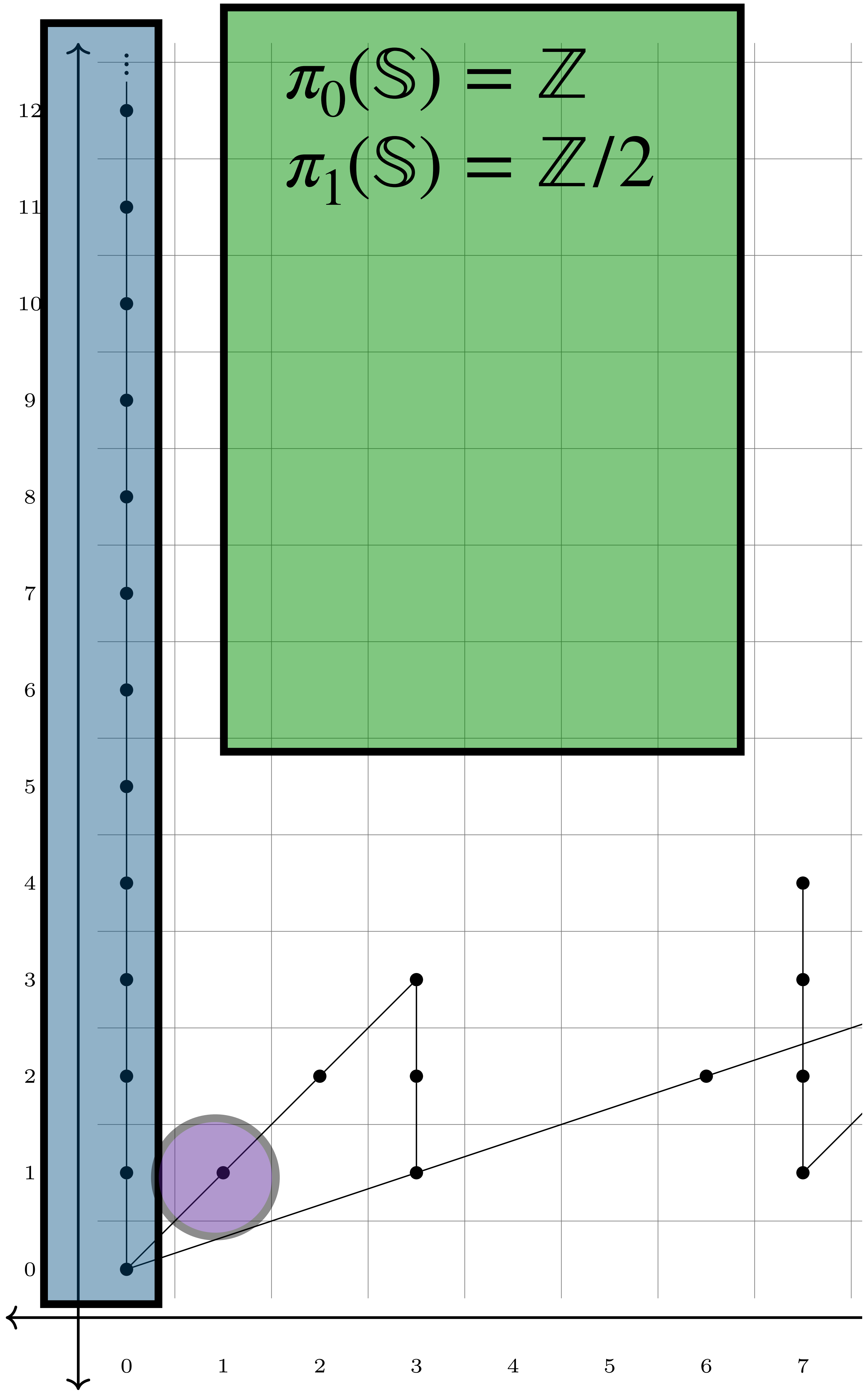
4
5
6
7

$$\pi_0(\mathbb{S}) = \mathbb{Z}$$
$$\pi_1(\mathbb{S}) = \mathbb{Z}/2$$



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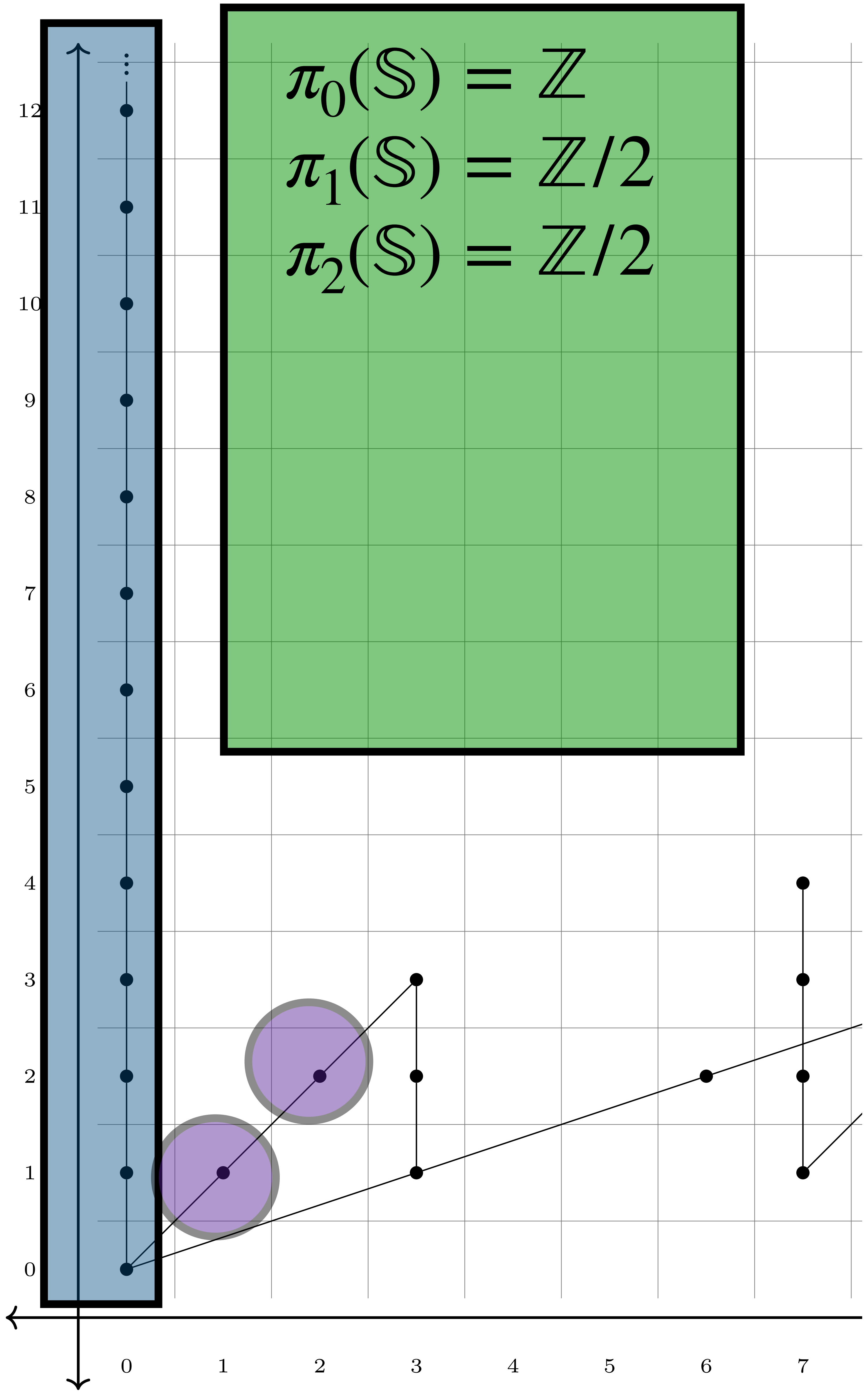
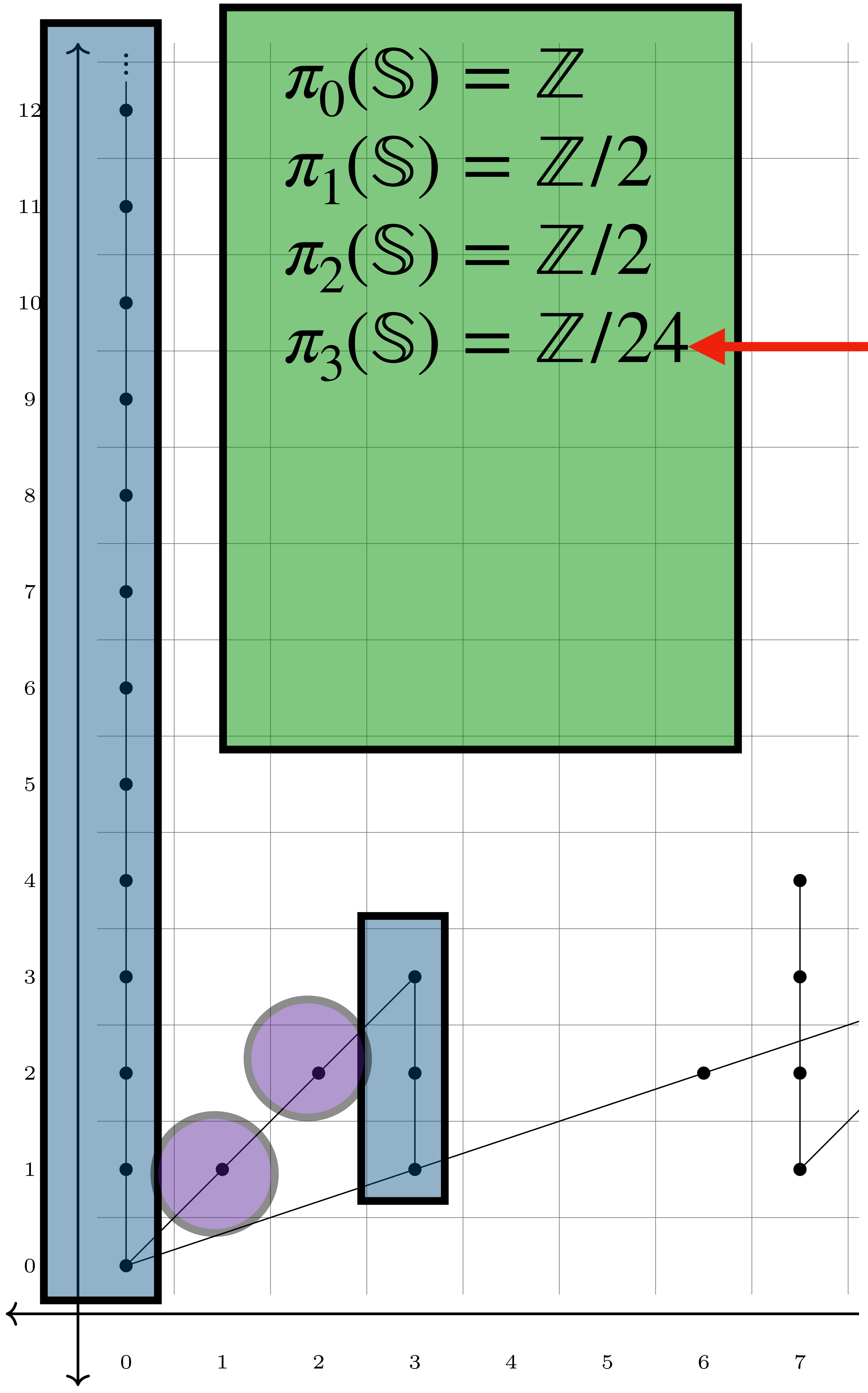
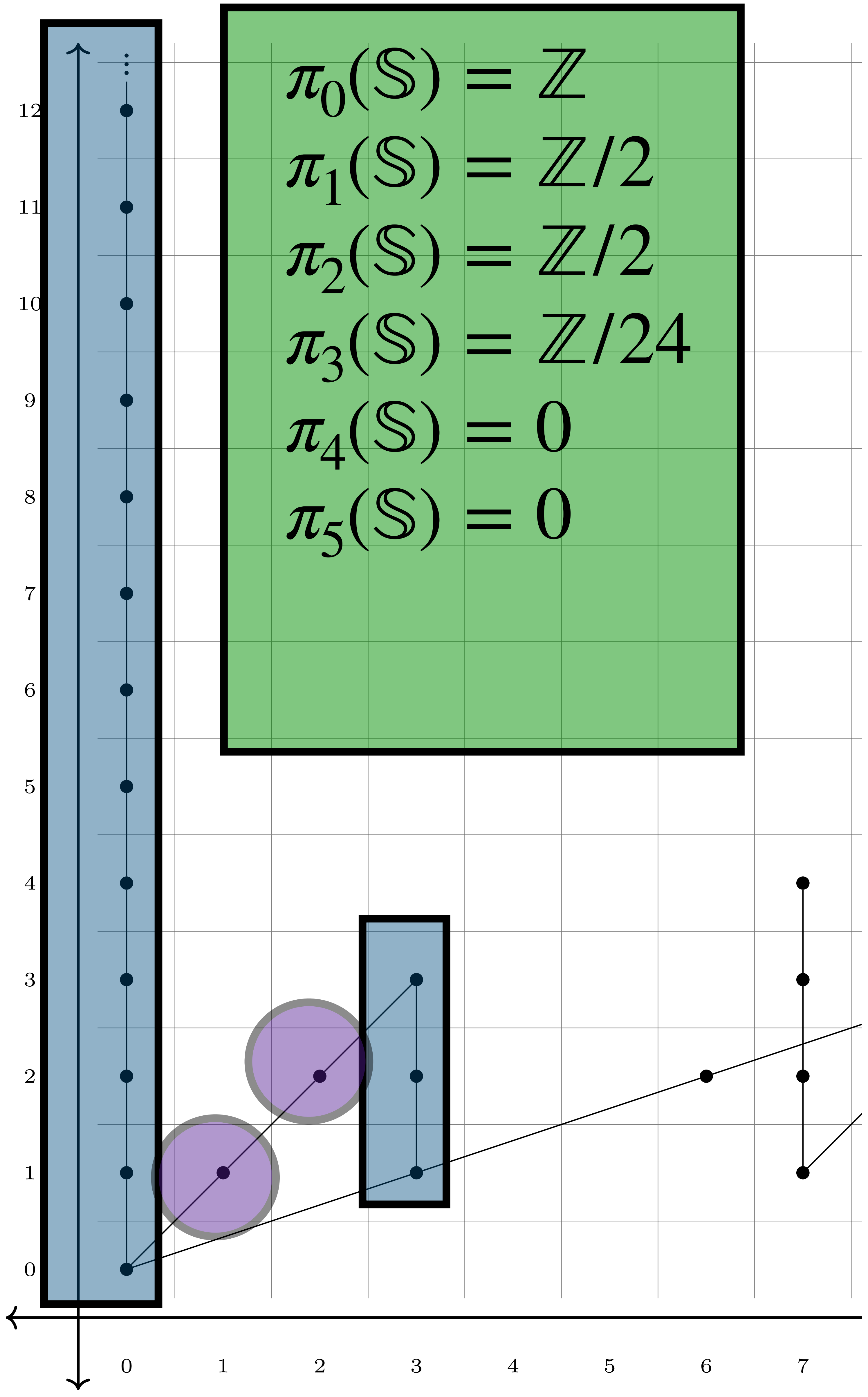




$$\pi_0(\mathbb{S}) = \mathbb{Z}$$
$$\pi_1(\mathbb{S}) = \mathbb{Z}/2$$
$$\pi_2(\mathbb{S}) = \mathbb{Z}/2$$
$$\pi_3(\mathbb{S}) = \mathbb{Z}/24$$

Missing a $\mathbb{Z}/3$!



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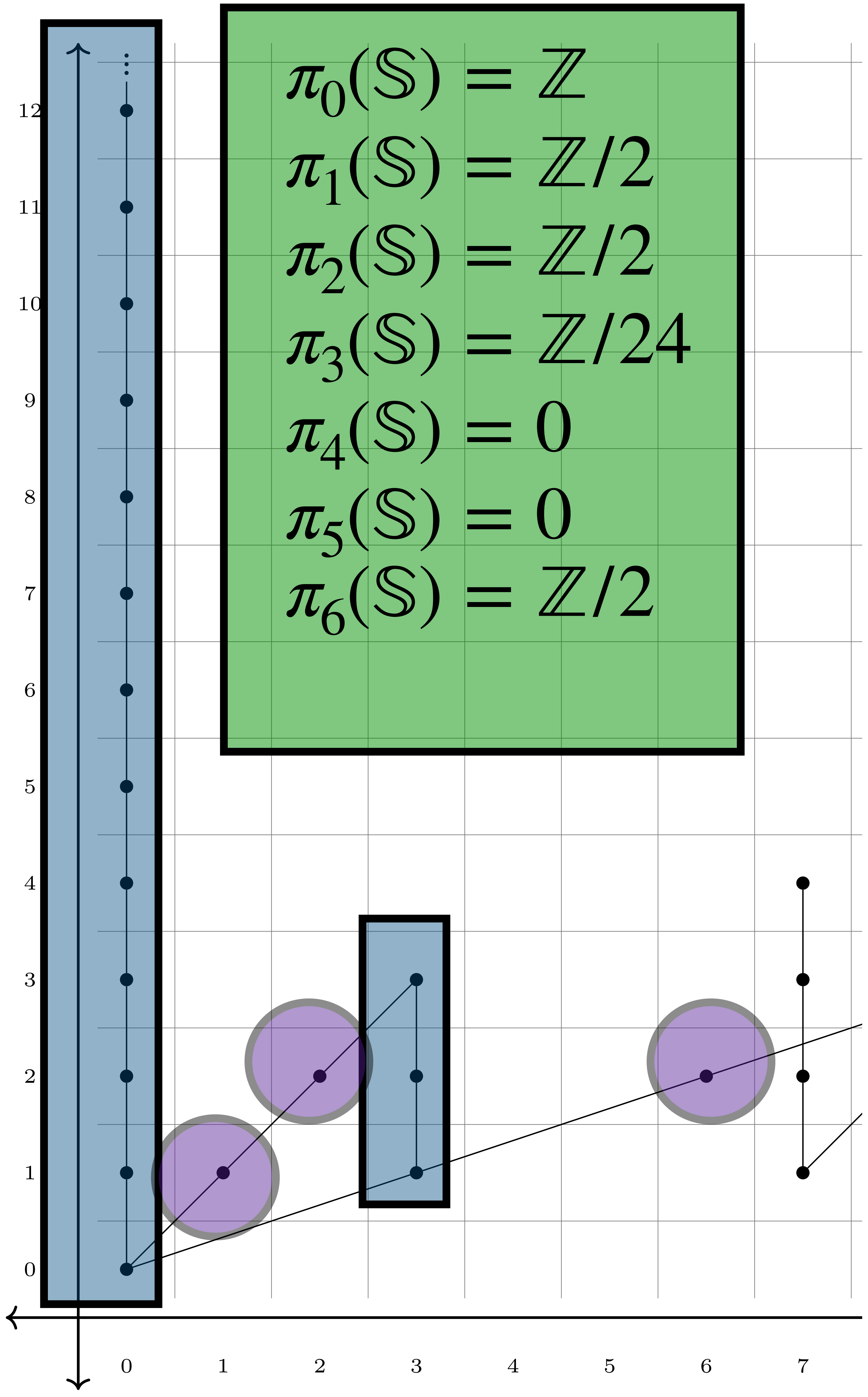
$$\pi_4(\mathbb{S}) = 0$$
$$\pi_5(\mathbb{S}) = 0$$



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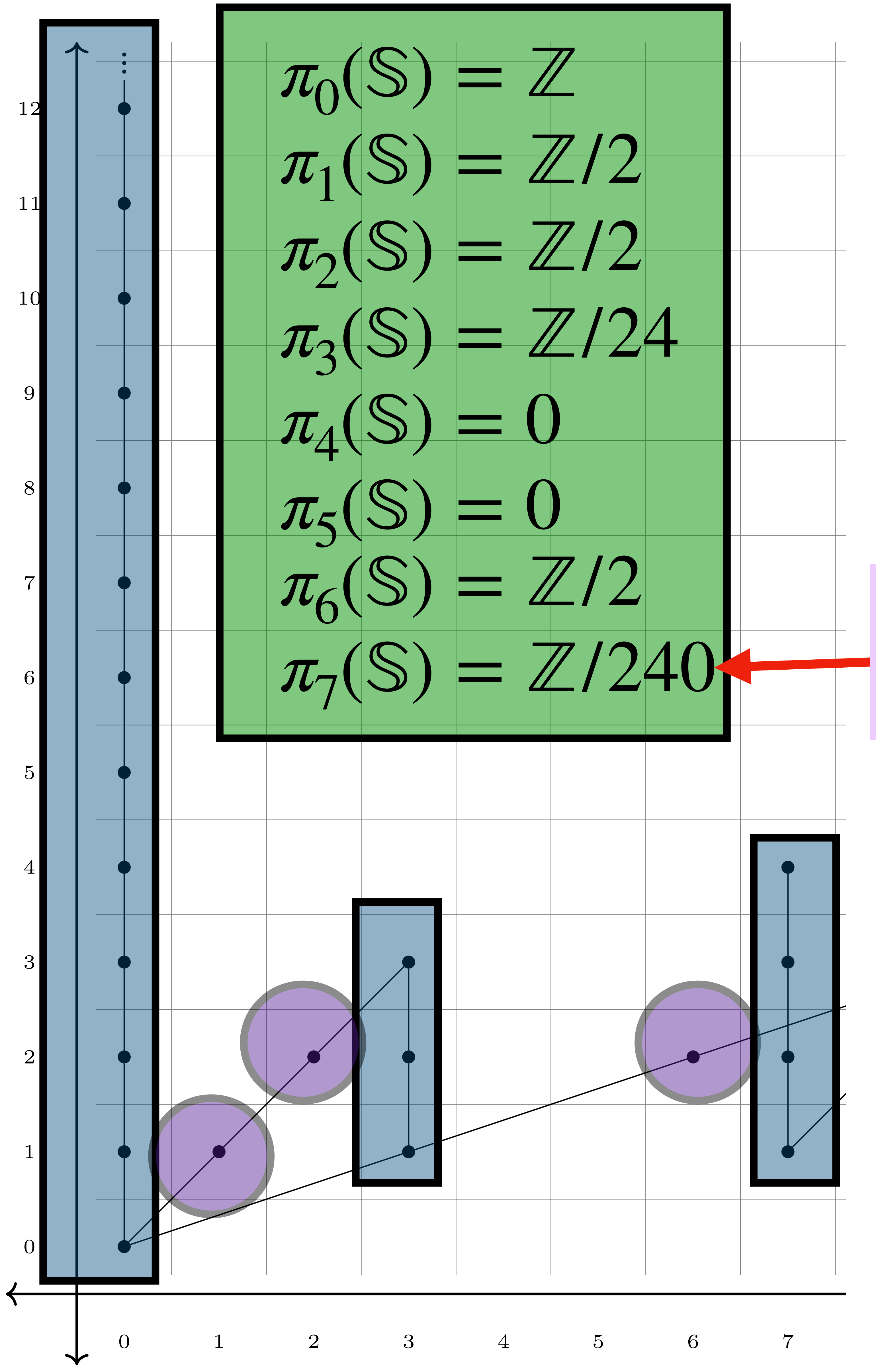

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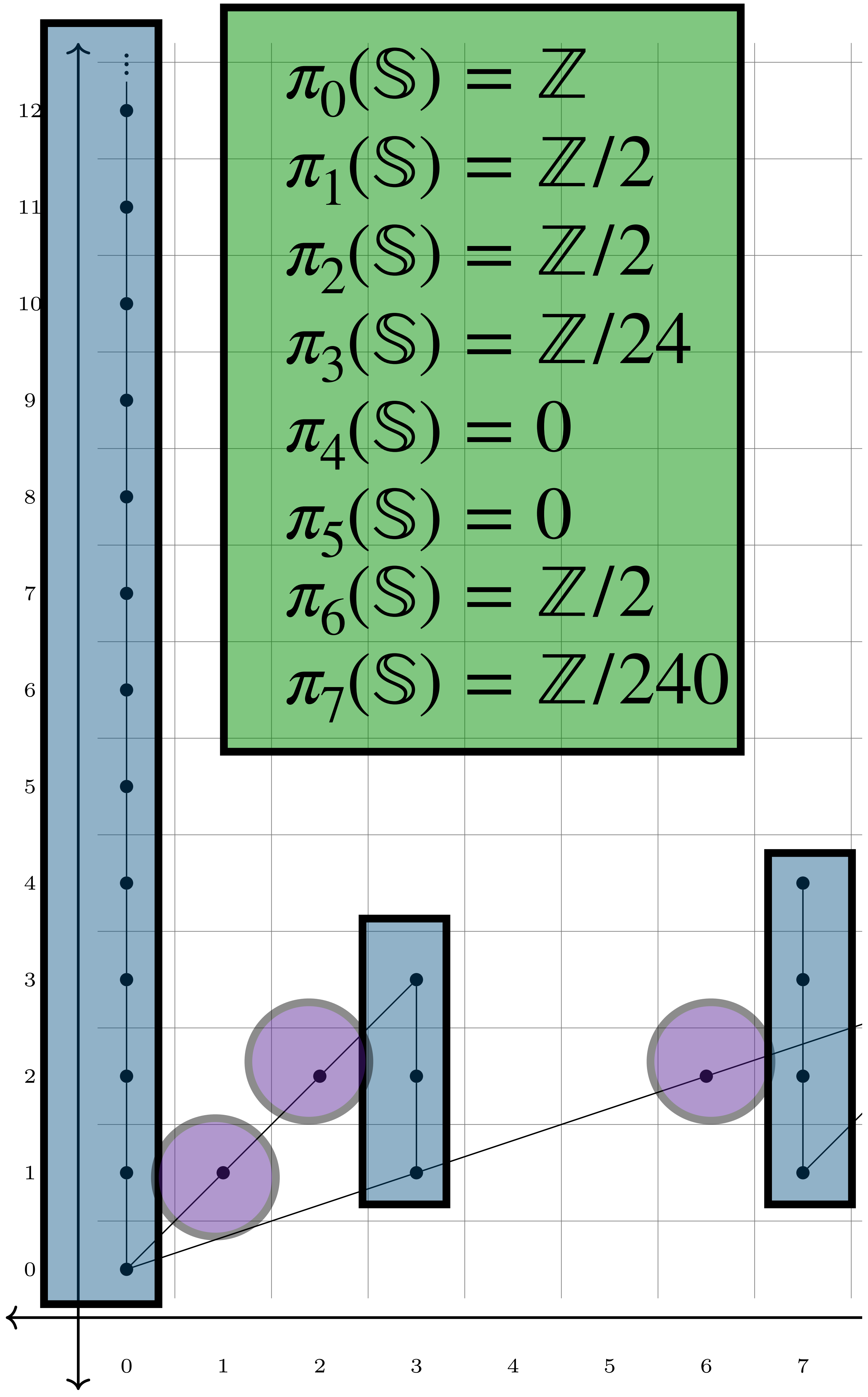
$\pi_6(\mathbb{S}) = \mathbb{Z}/2$

$\pi_7(\mathbb{S}) = \mathbb{Z}/240$

Missing a $\mathbb{Z}/15$!



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What about the rest?

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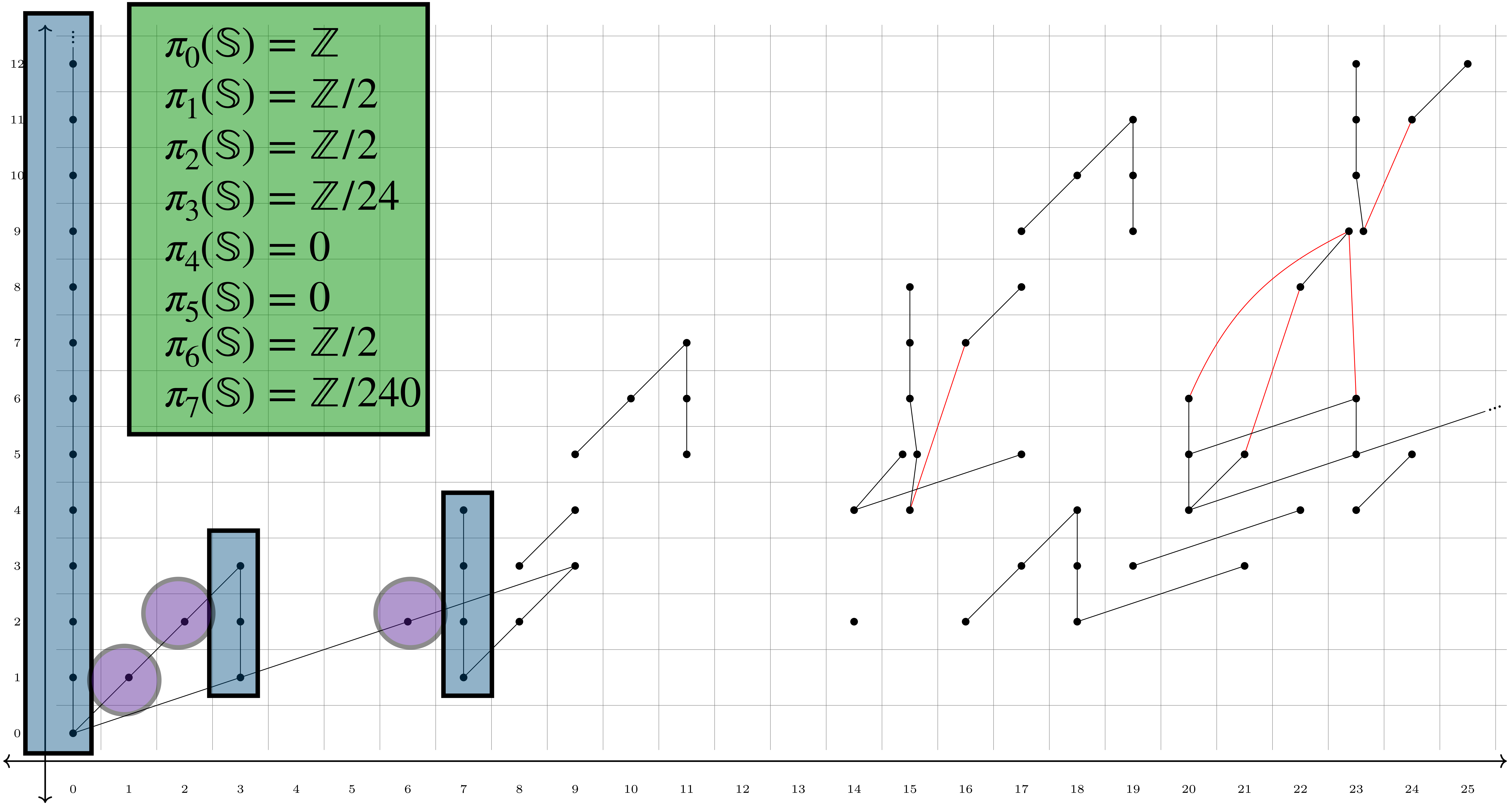
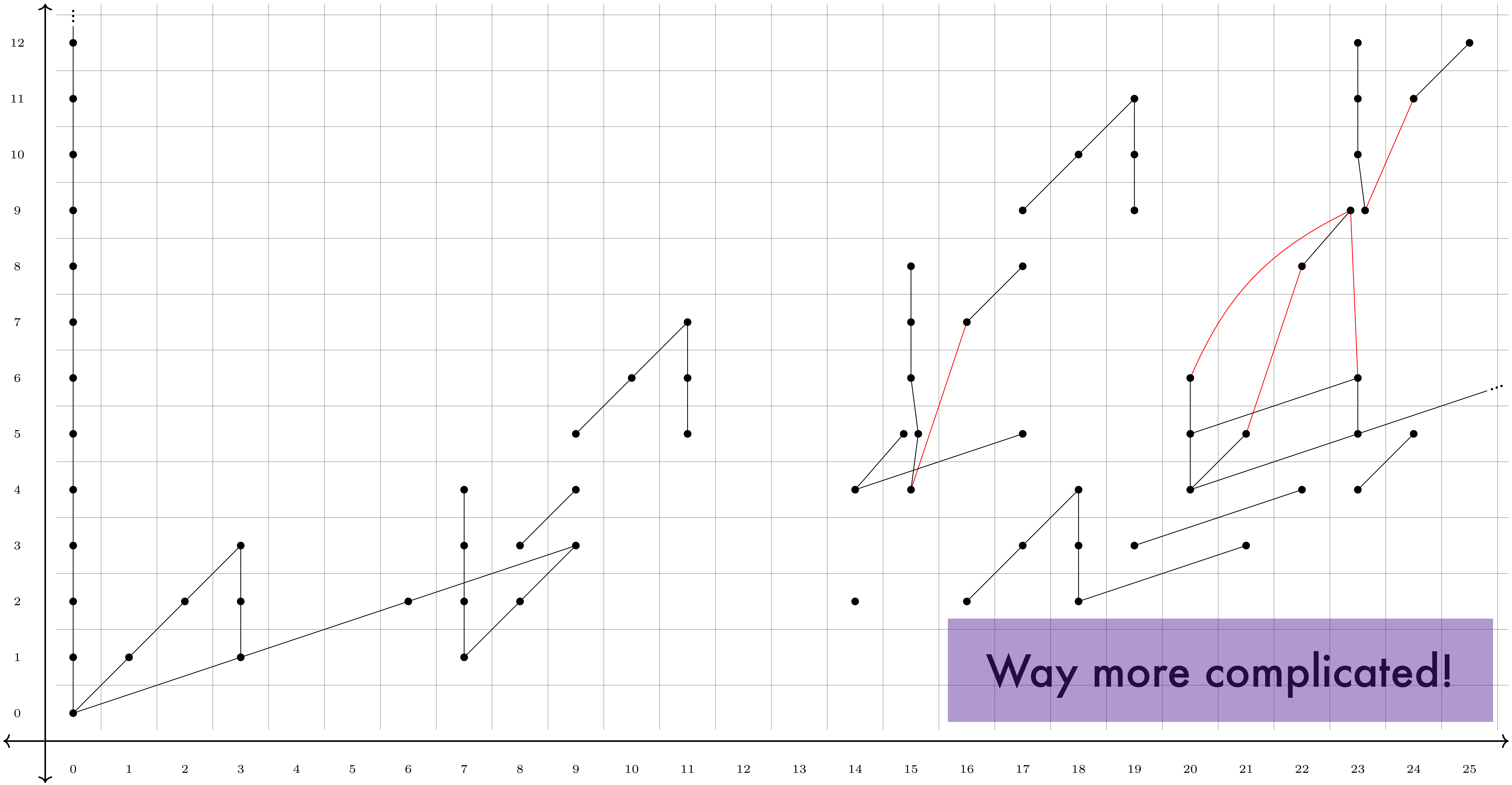
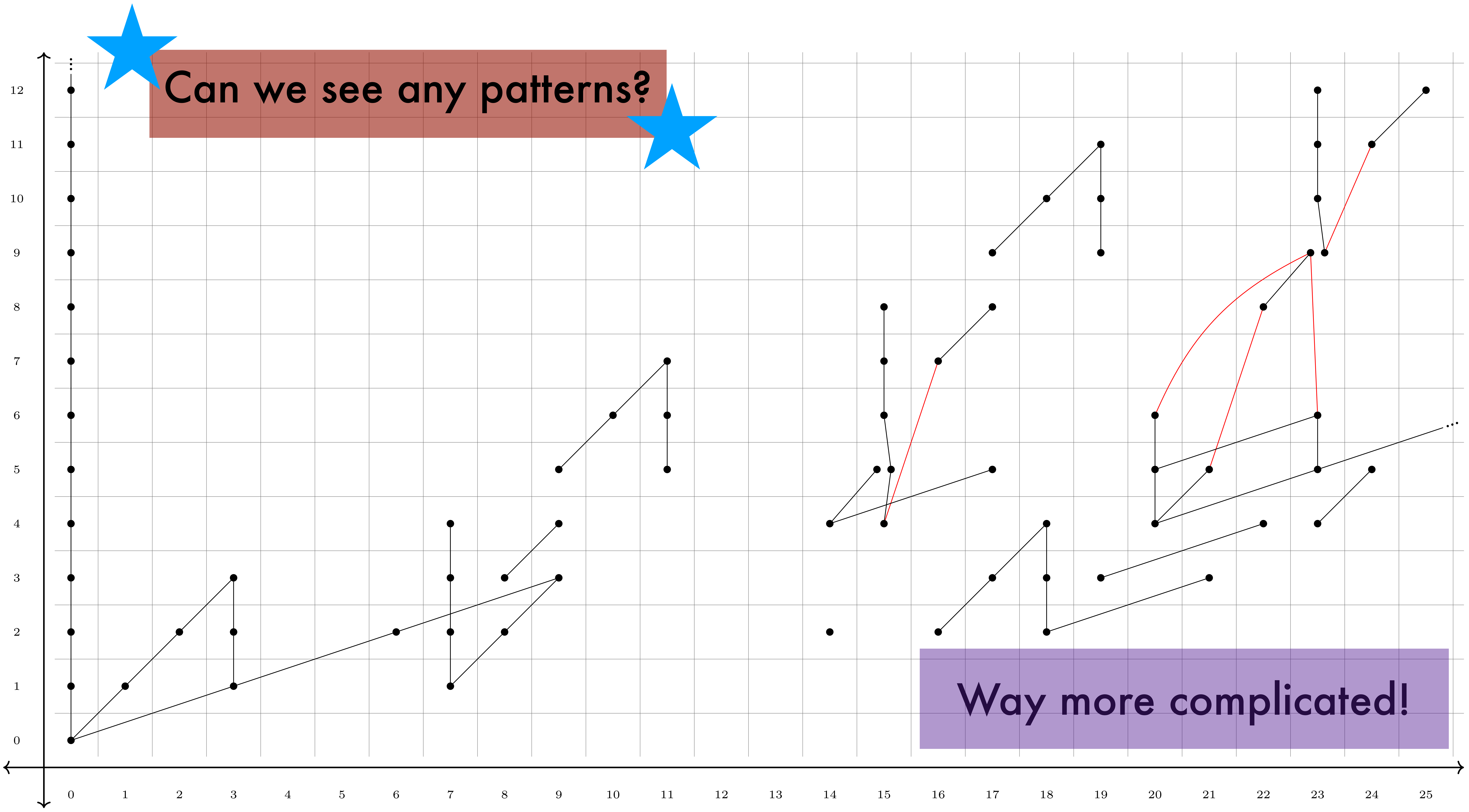



0 1 2 3 4 5 6 7 8 9 10 11 12
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
Way more complicated!


Can we see any patterns?
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0
1
2
3
4
5
6
7
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16
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25

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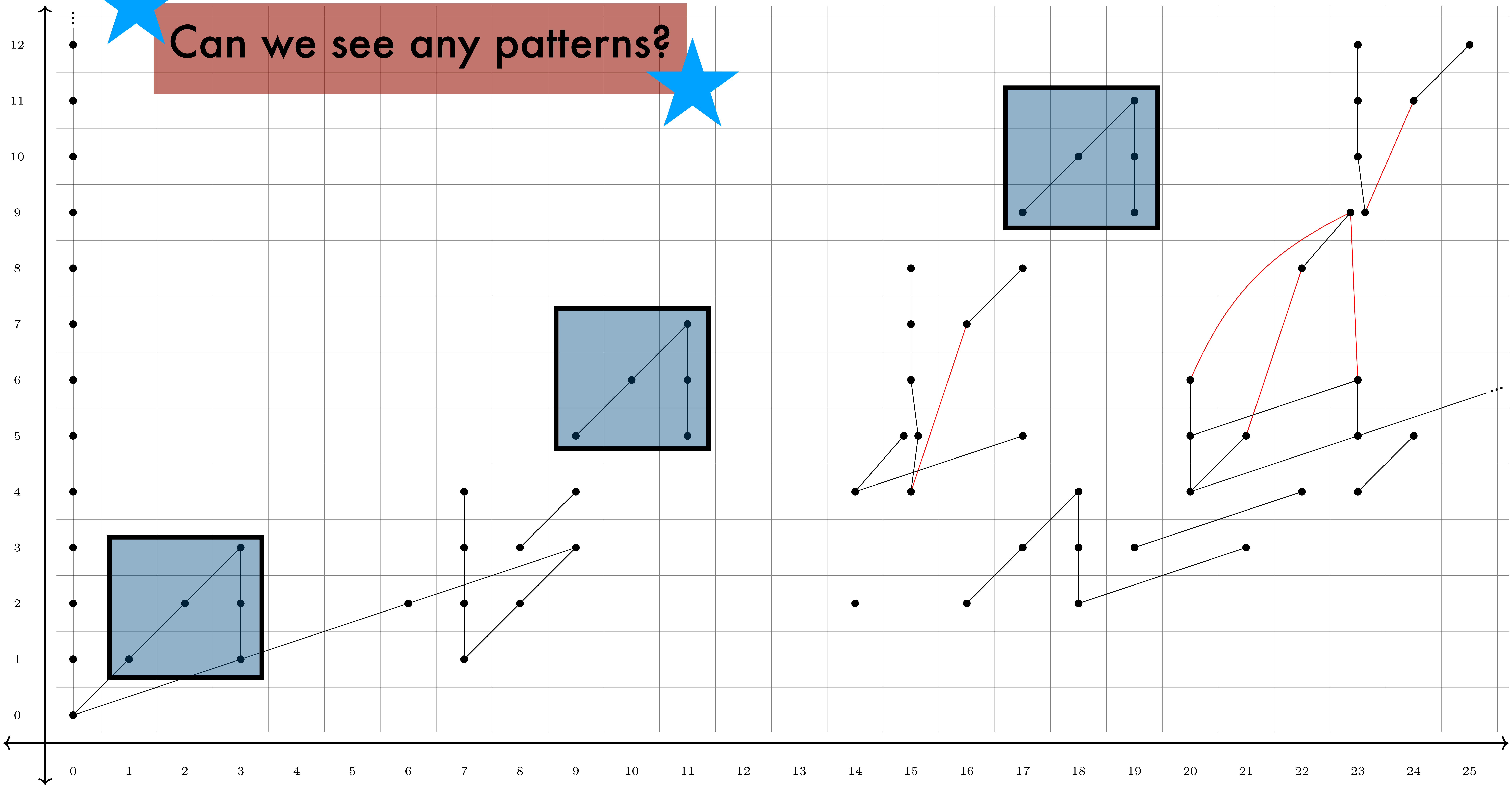

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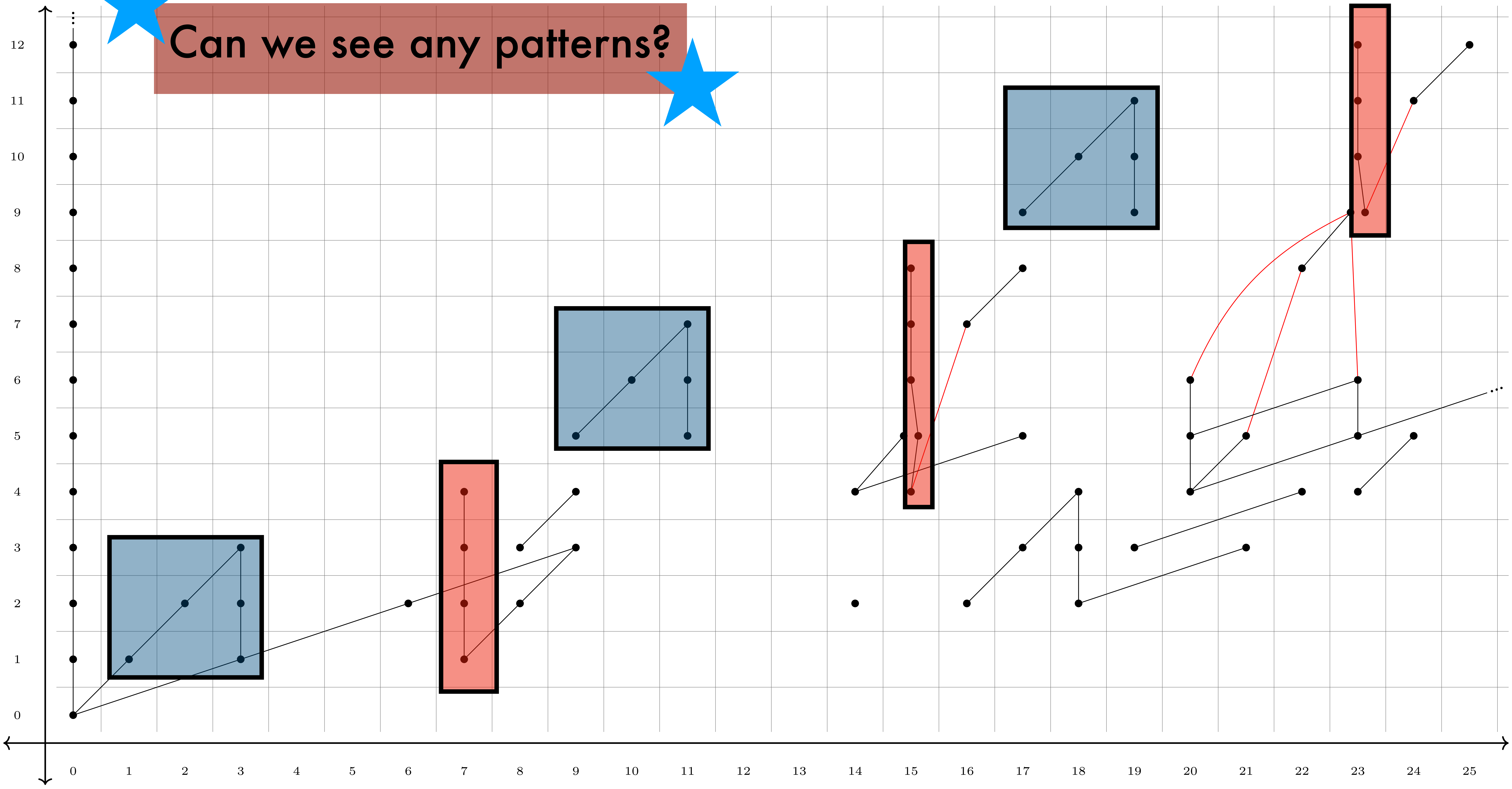

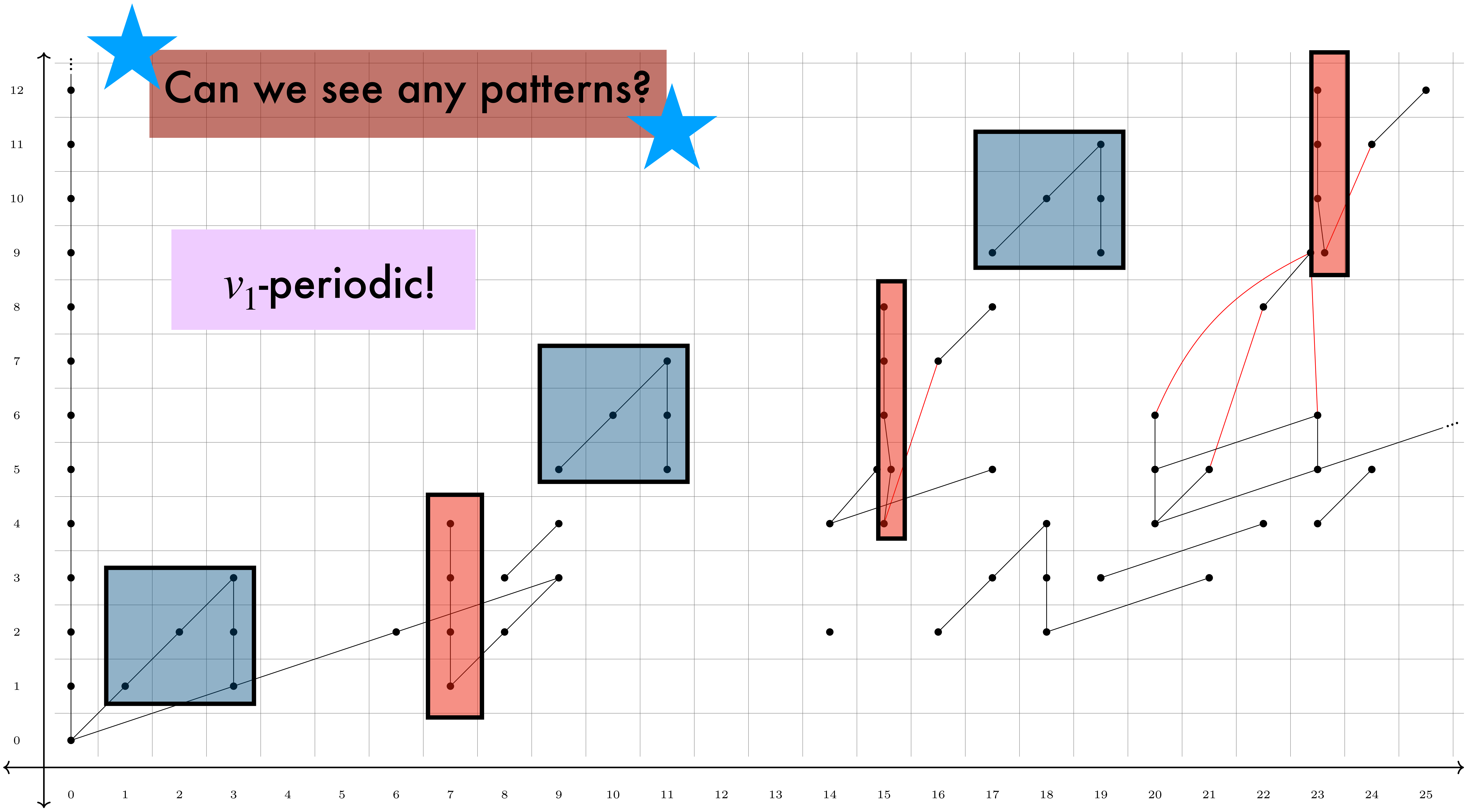

Can we see any patterns?
v_1-periodic!
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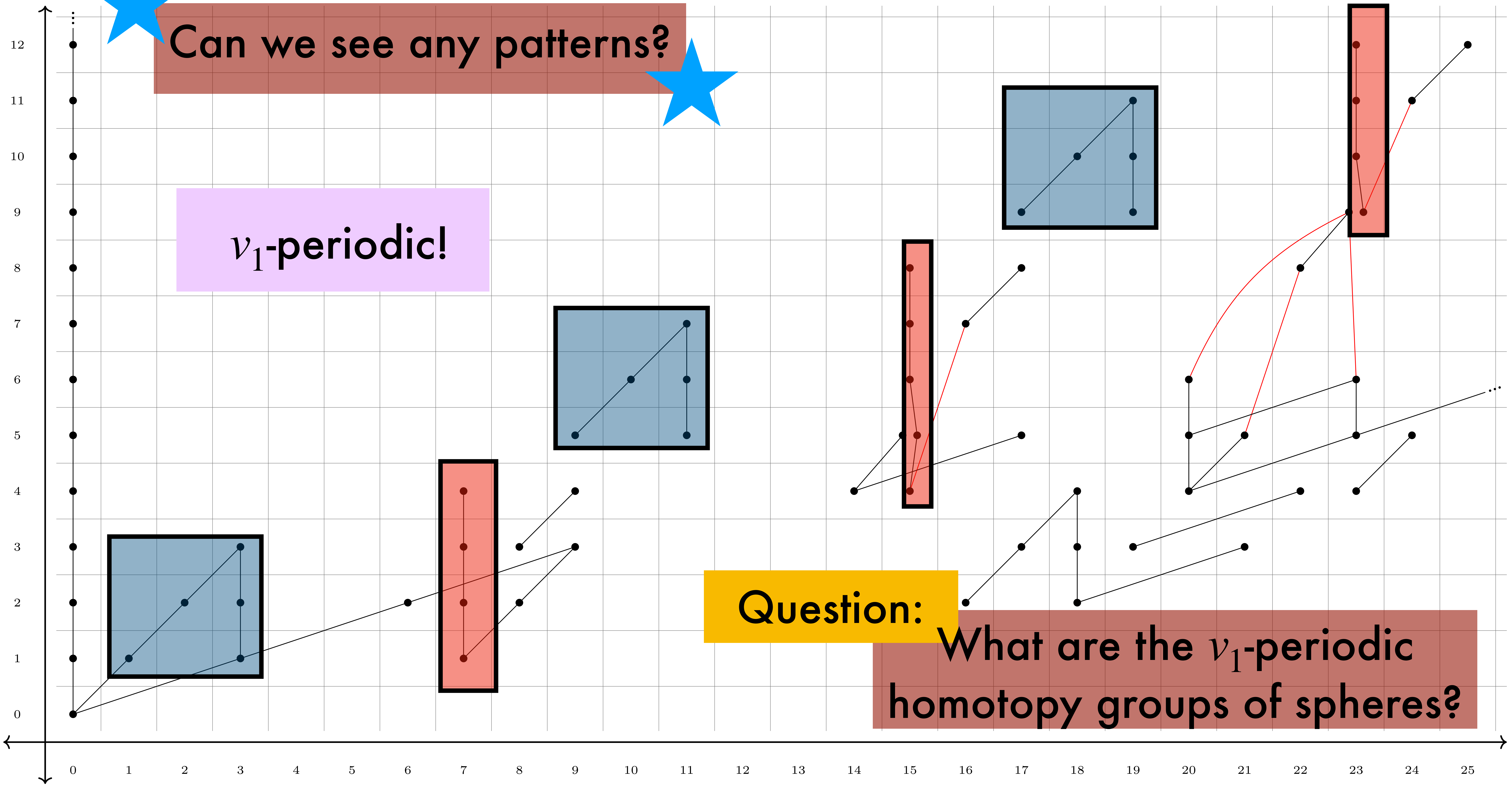

Question: What are the v_1-periodic homotopy groups of spheres?

I lied a little.

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Method: a different Adams spectral sequence!

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$\pi_{**}^{F}(\mathbf{kq} \otimes \mathbf{kq})$

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+ Change of rings

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Reduced to computing $\mathrm{Ext}^{***}_{A(1)}(\mathbb{M}_2, B_0^F(k))$

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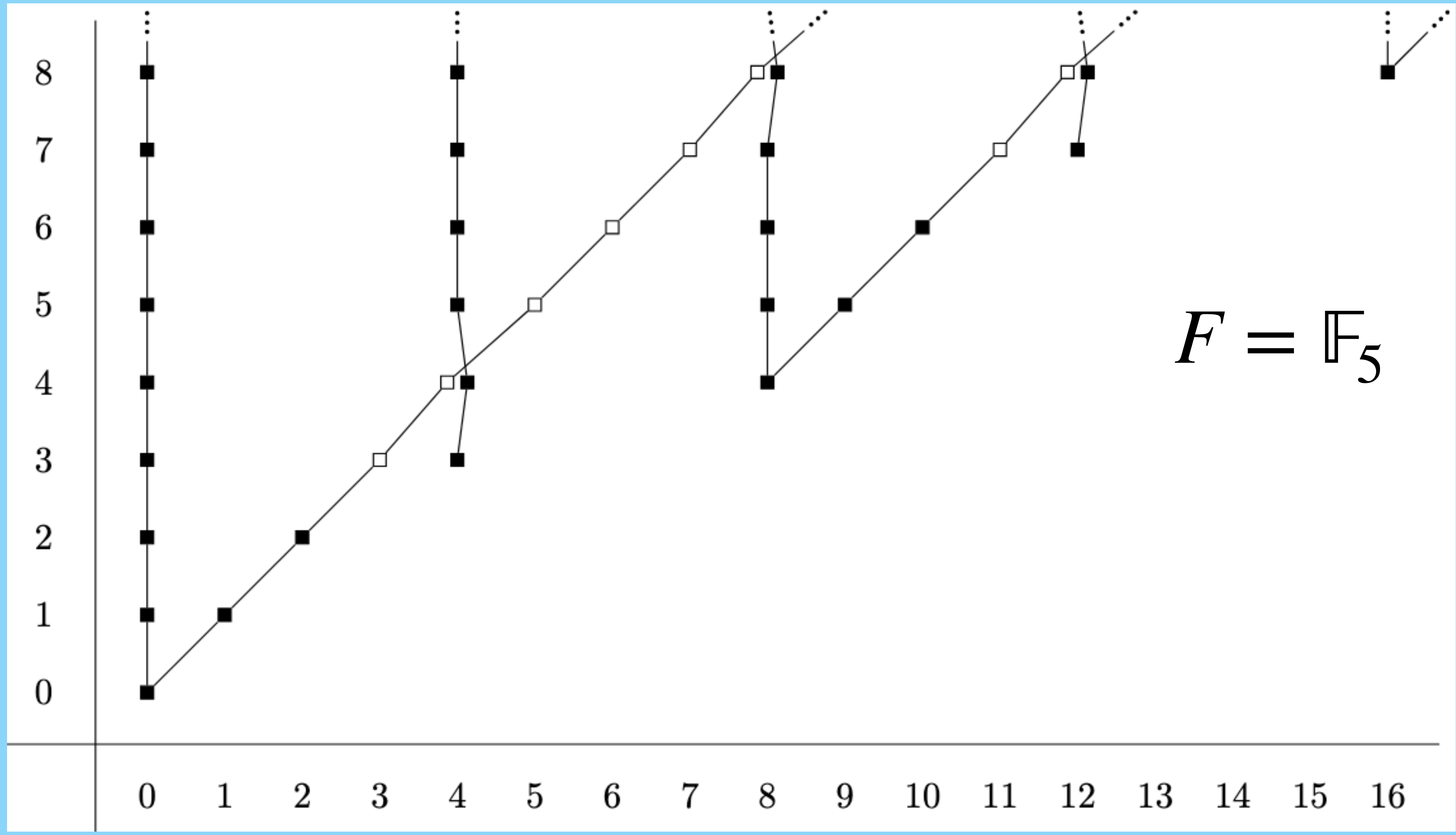

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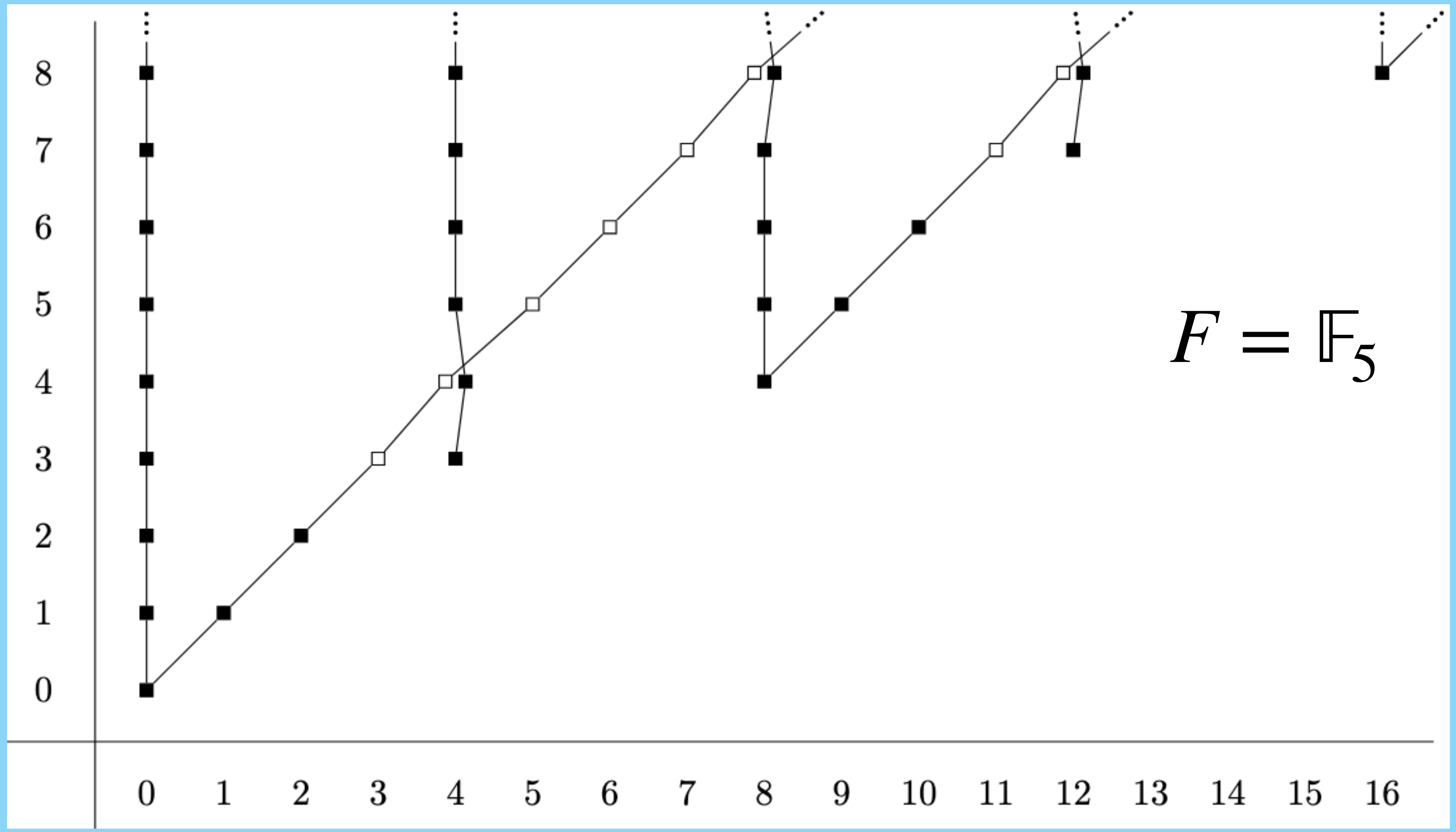

$F = \mathbb{F}_5$

A ■ denotes $\mathbb{F}_2[\tau, u]/(u^2)$

A □ denotes $\mathbb{F}_2[u]/(u^2)$

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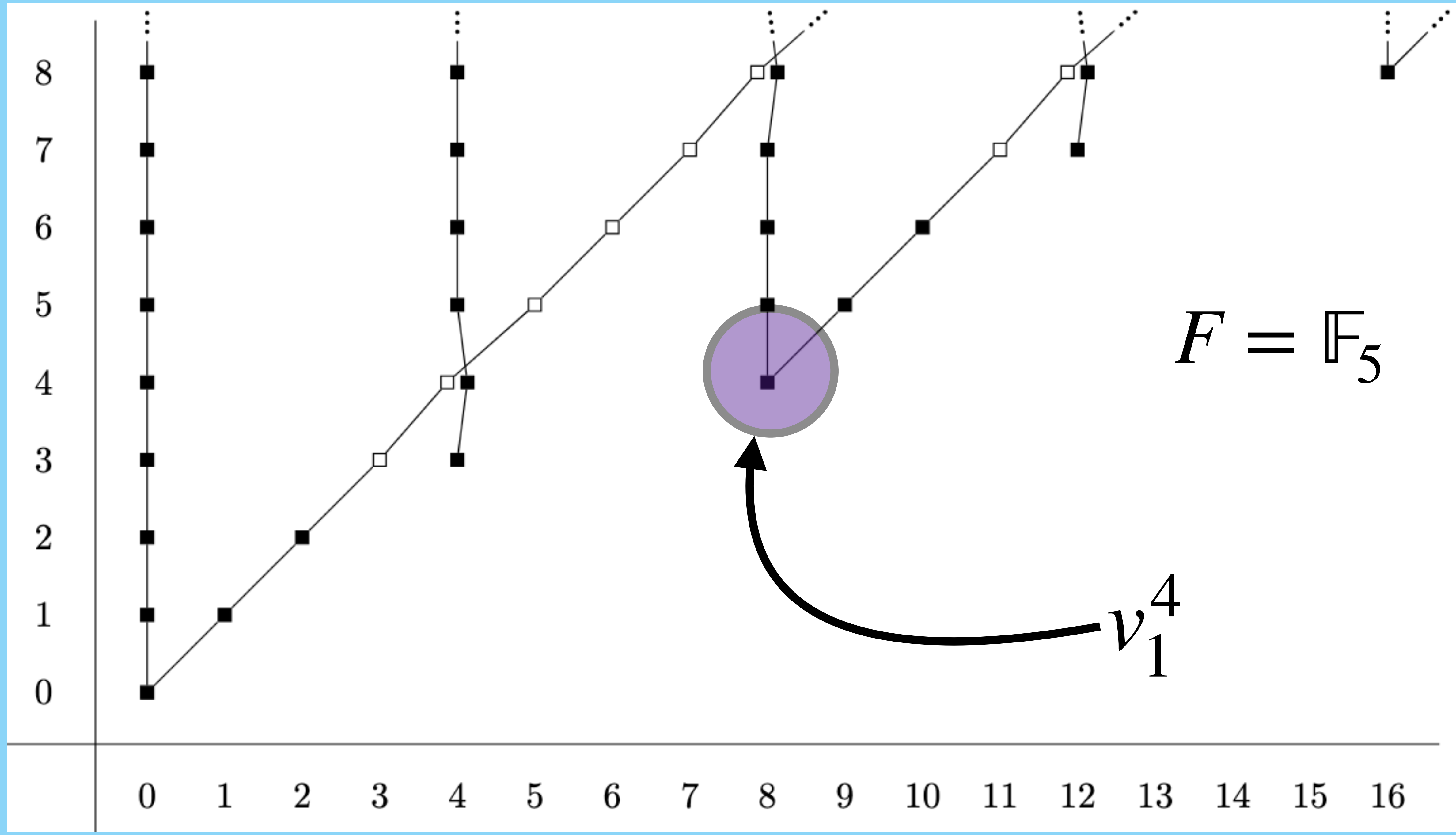

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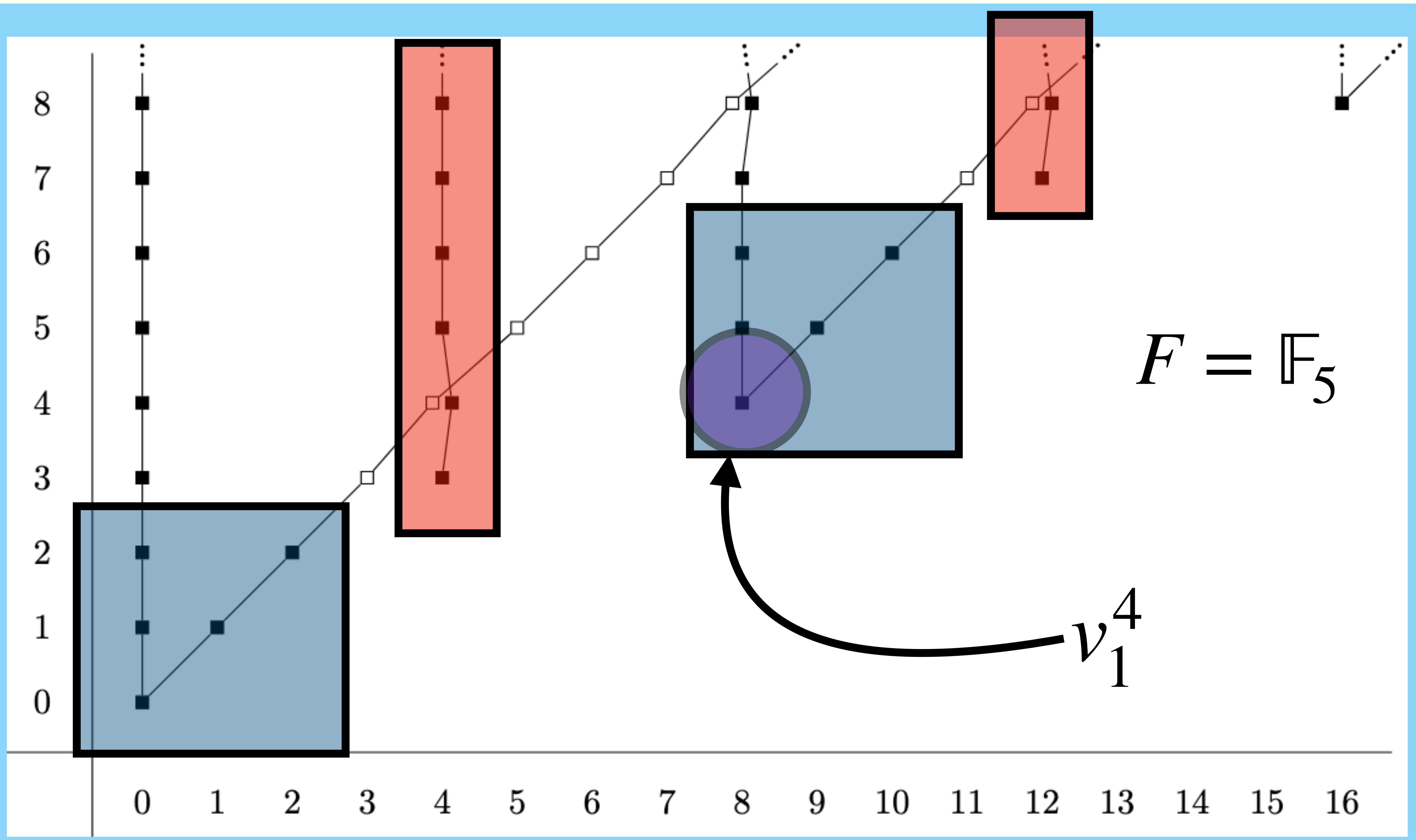

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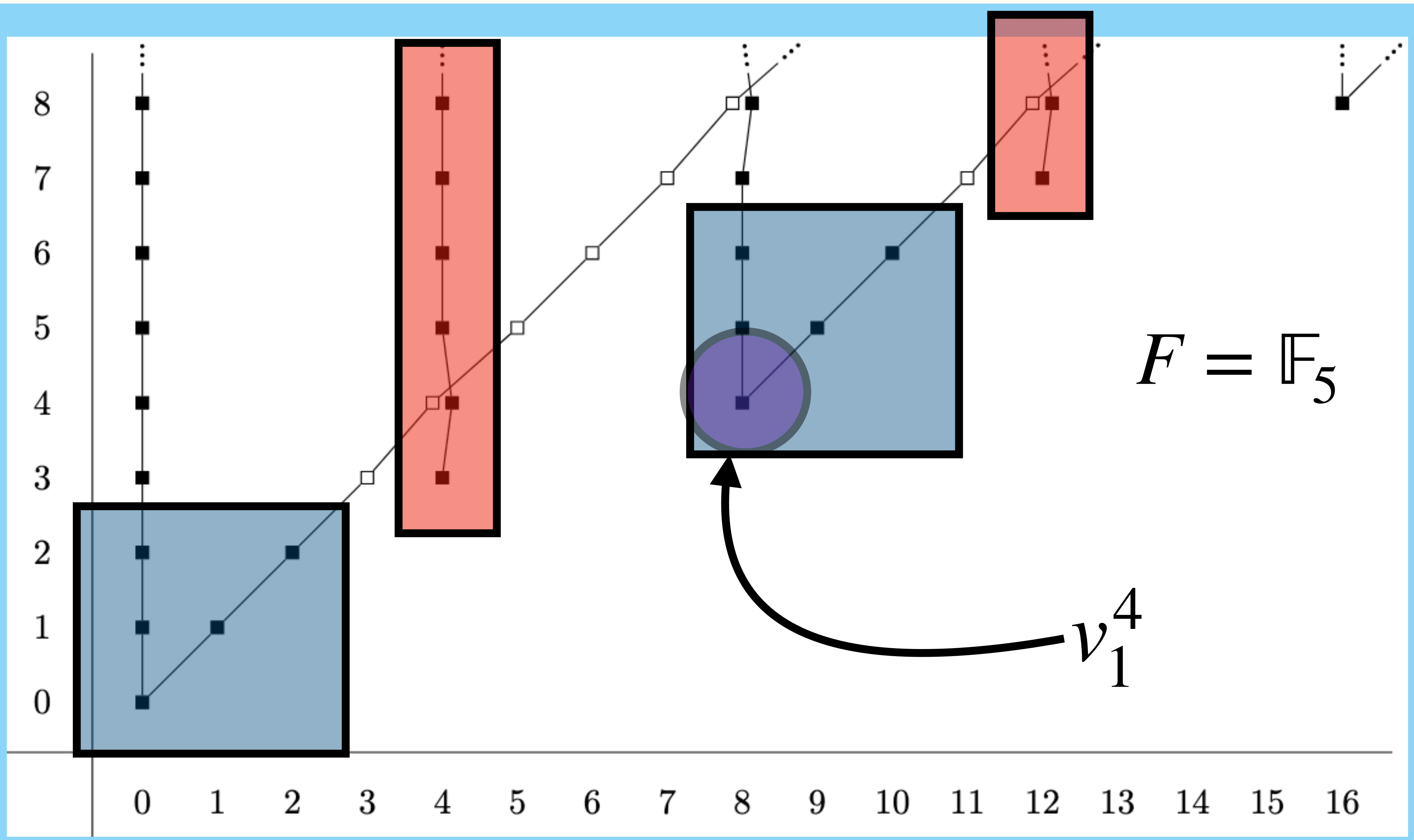

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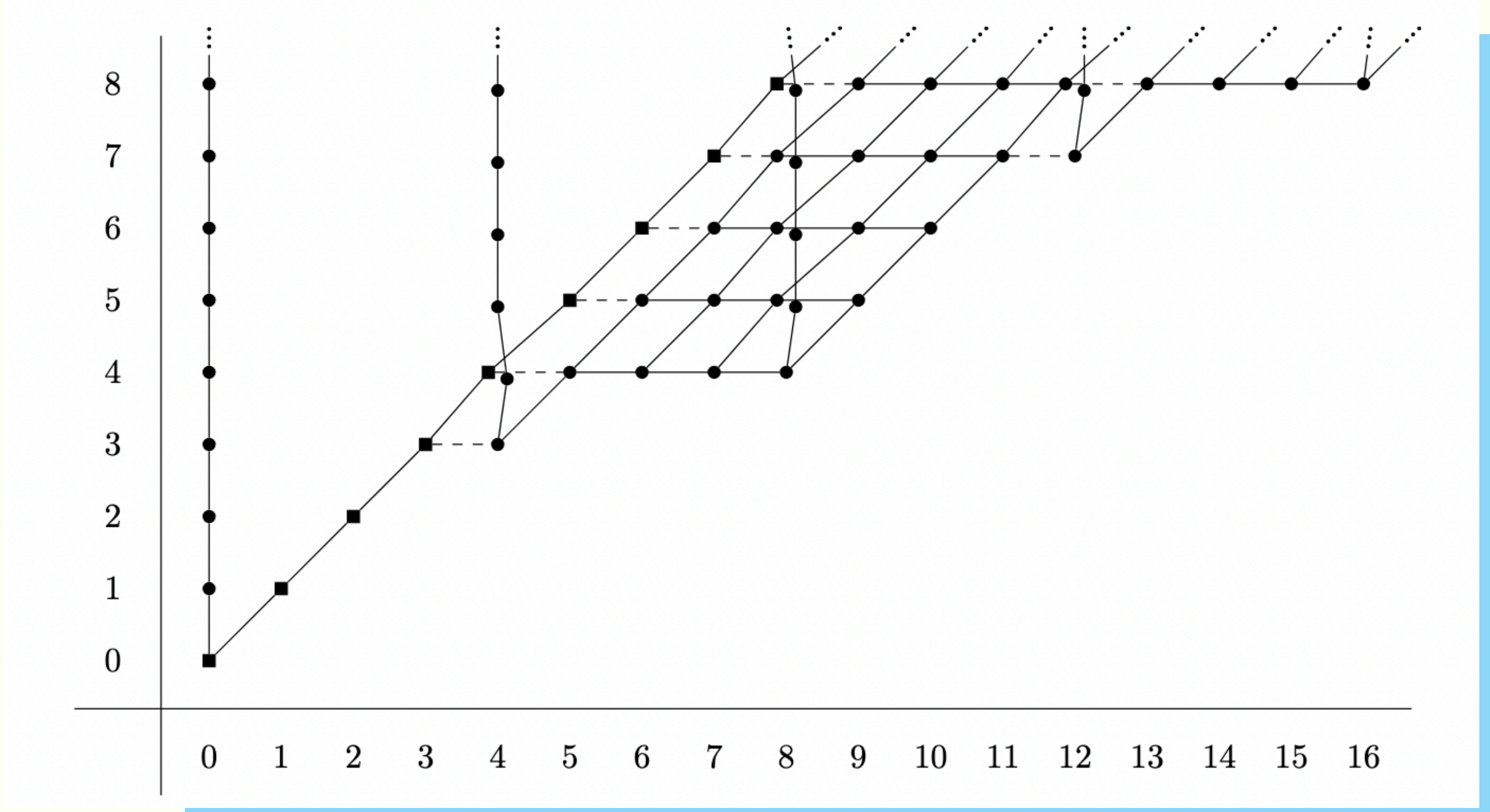

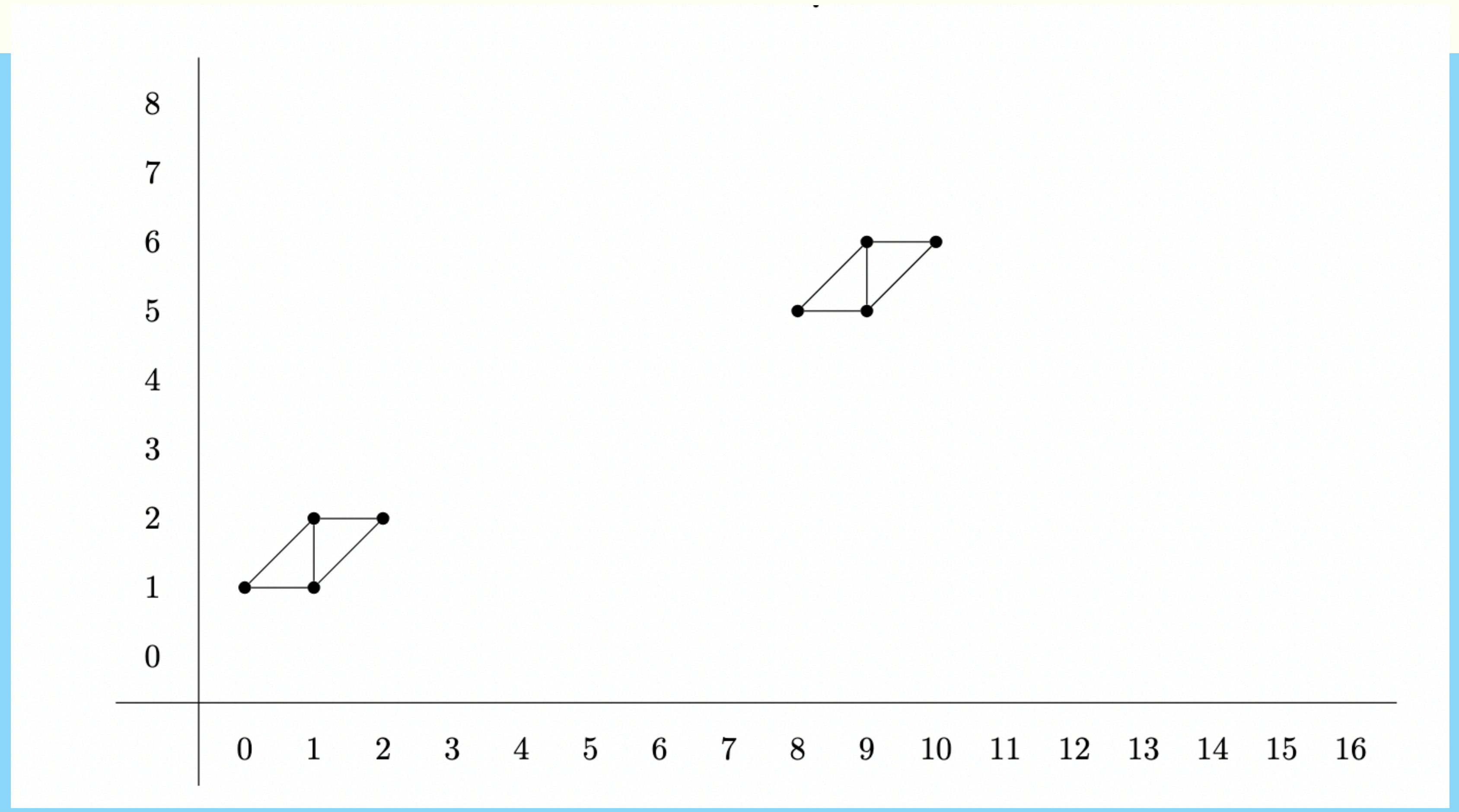

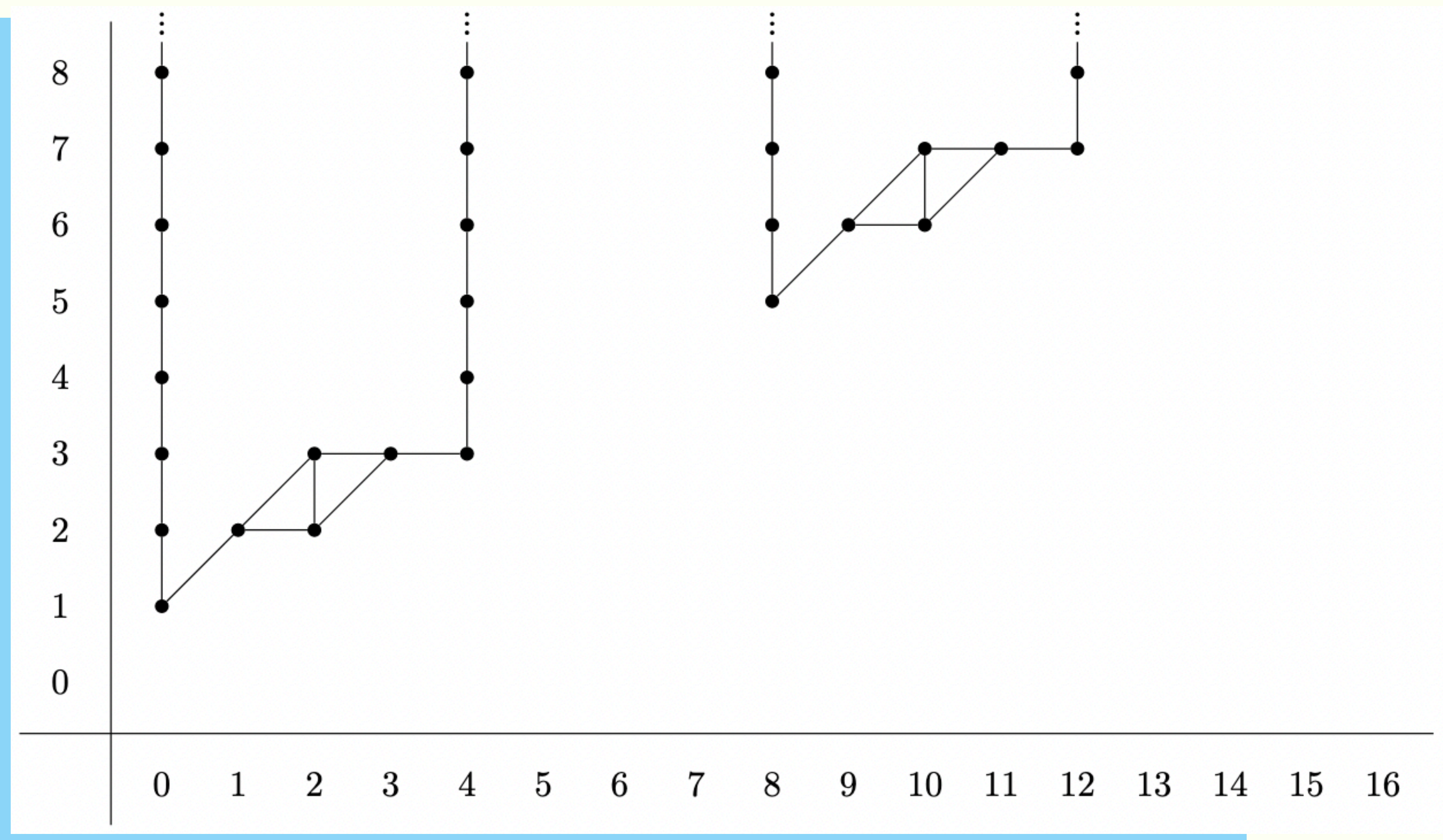

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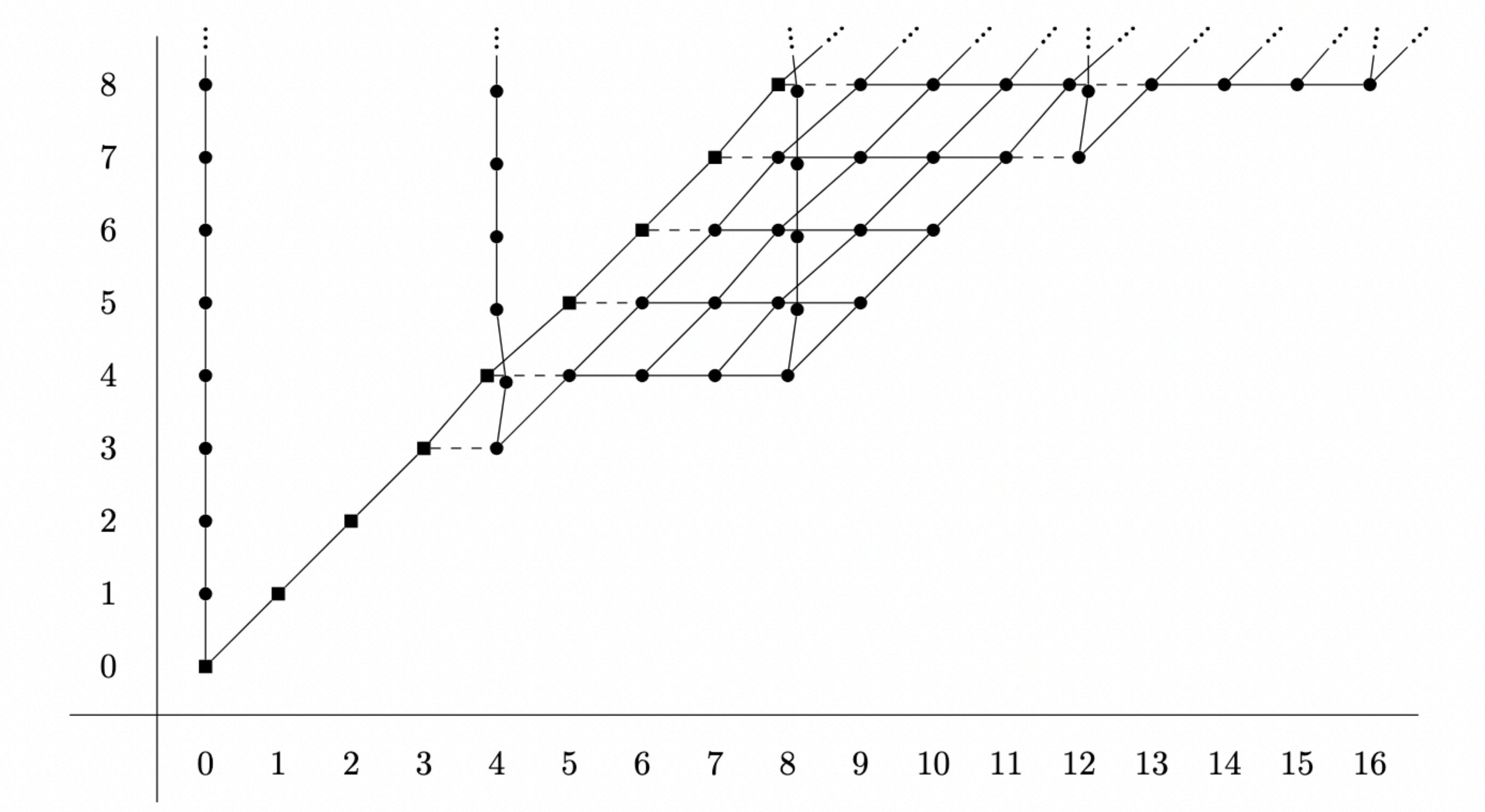

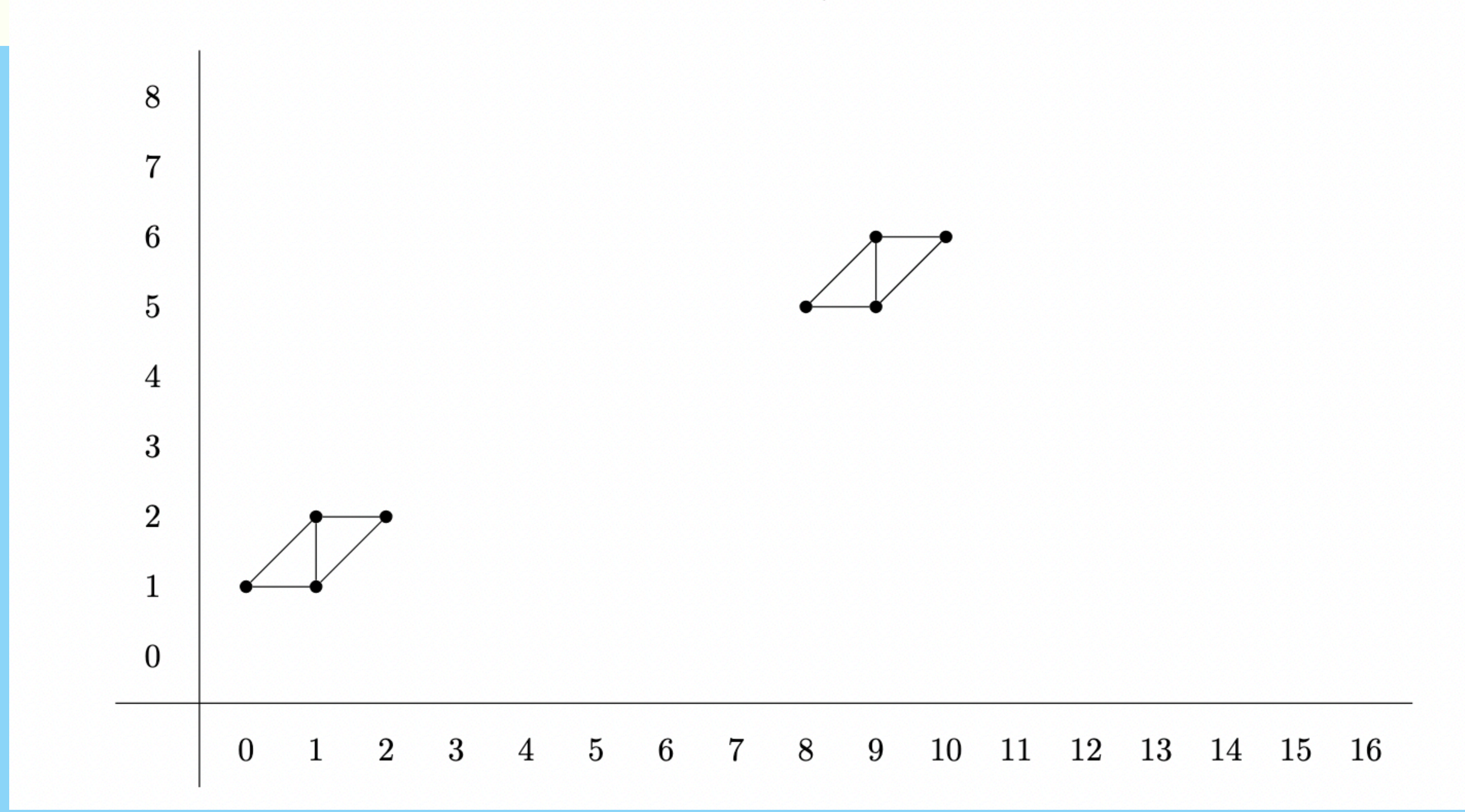

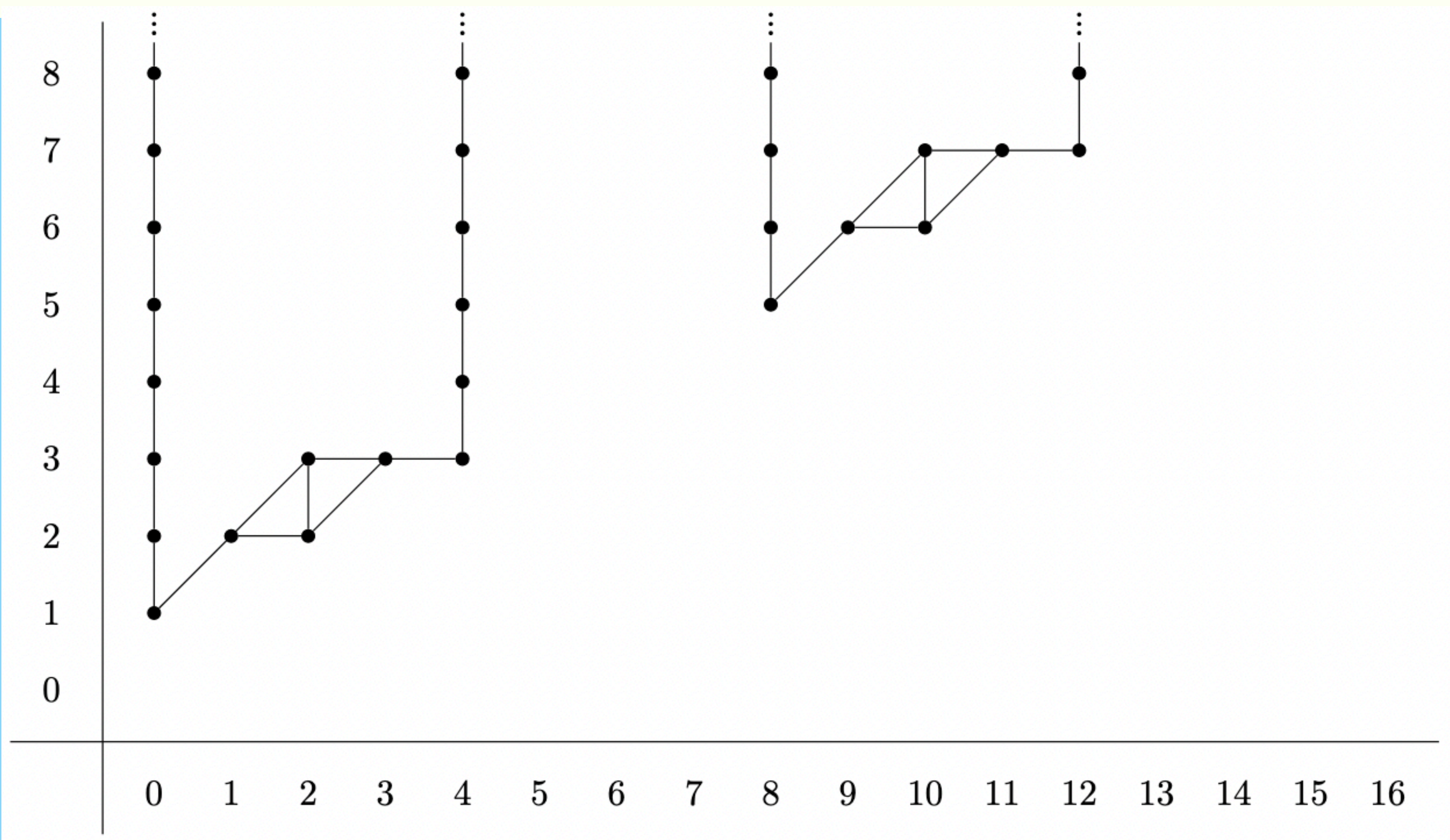

All overlayed!

Over $\mathbb{R}$...

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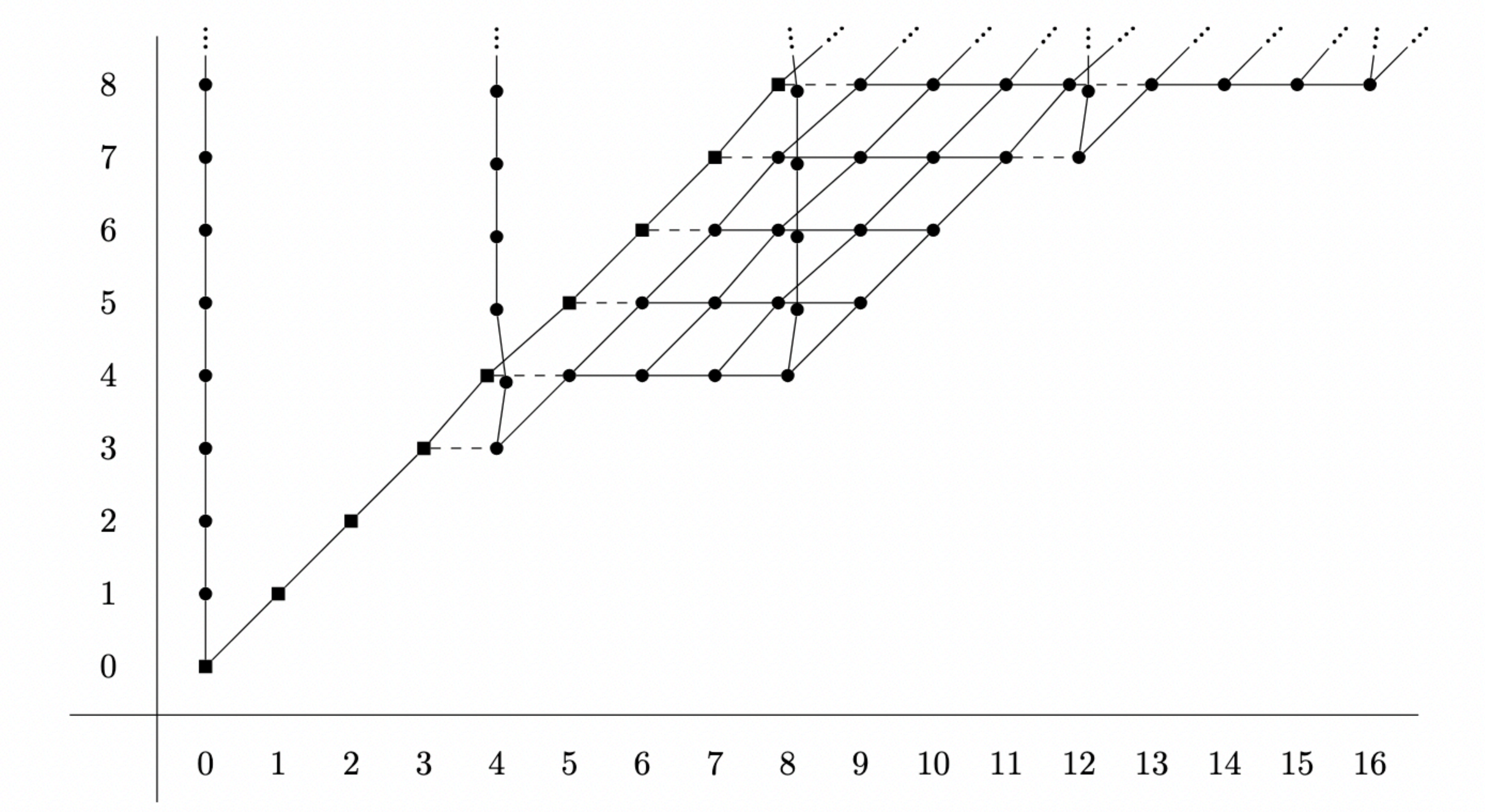
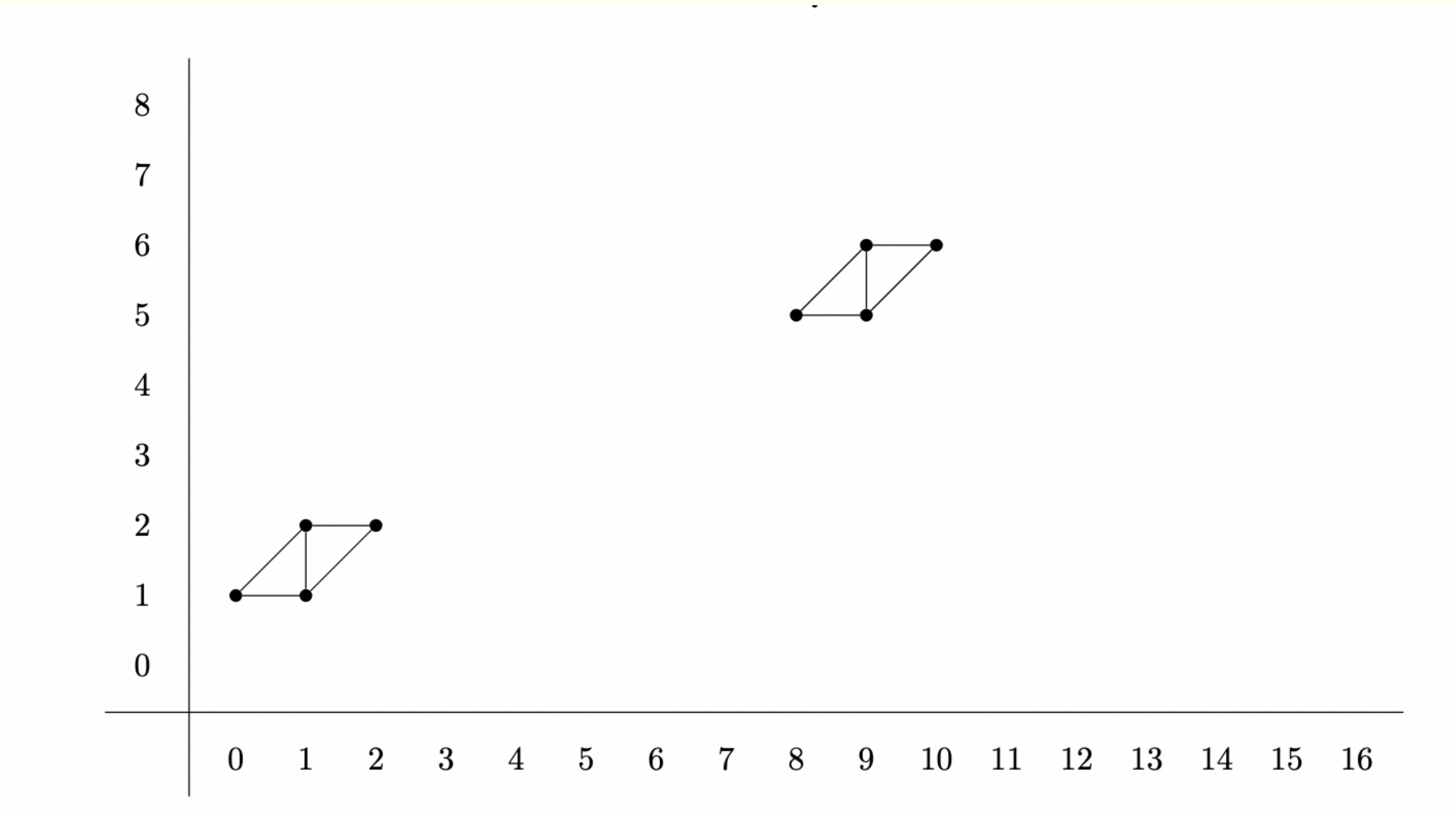
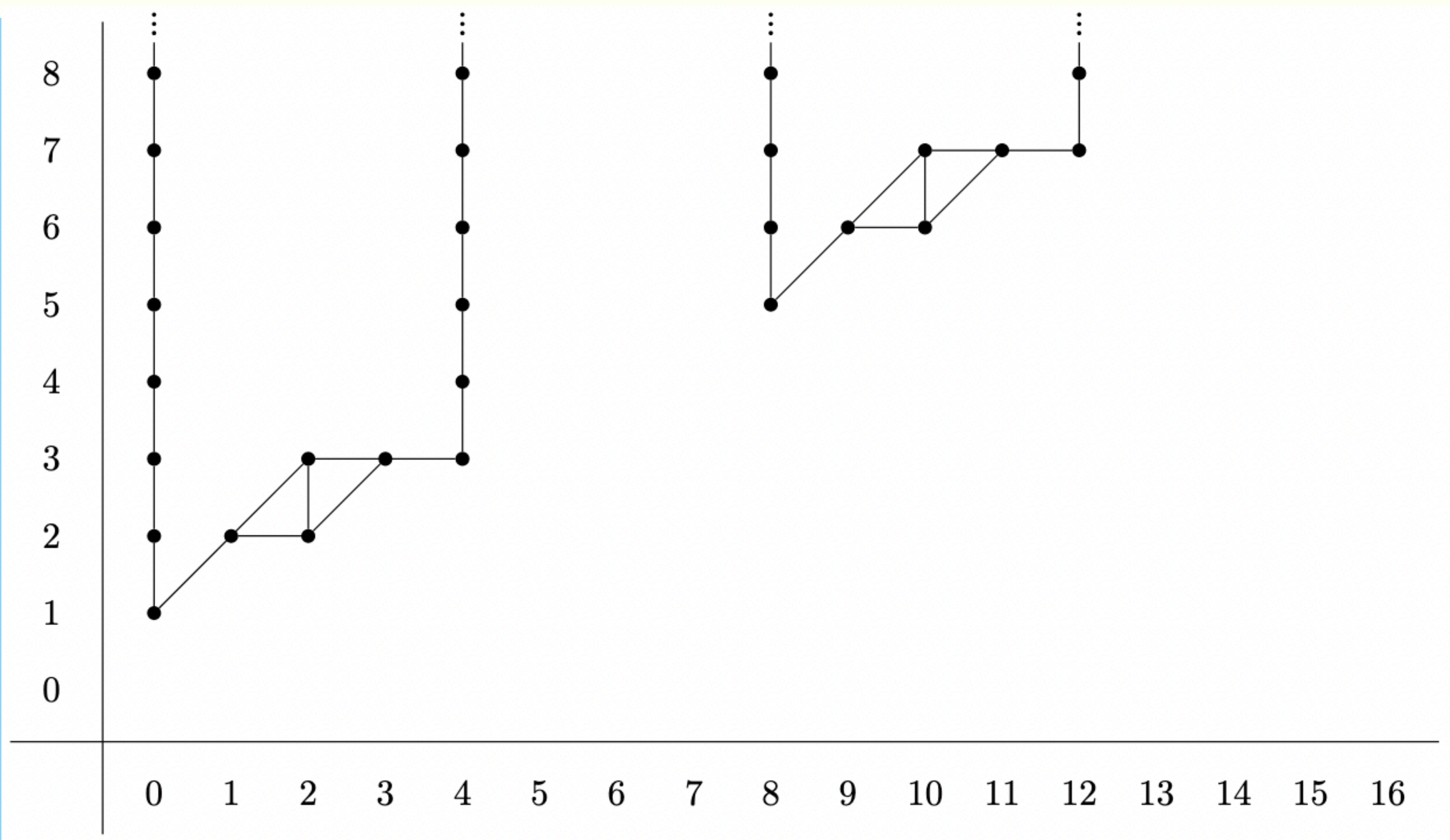
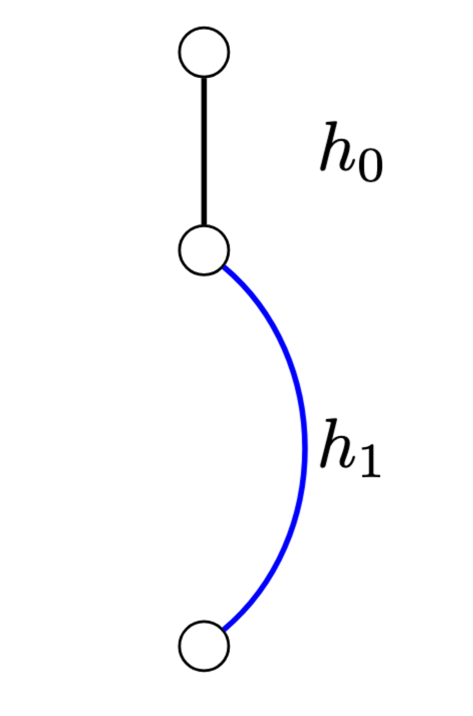






All overlayed!

$B_0^F(1) \cong$ 

Over $\mathbb{R}$...

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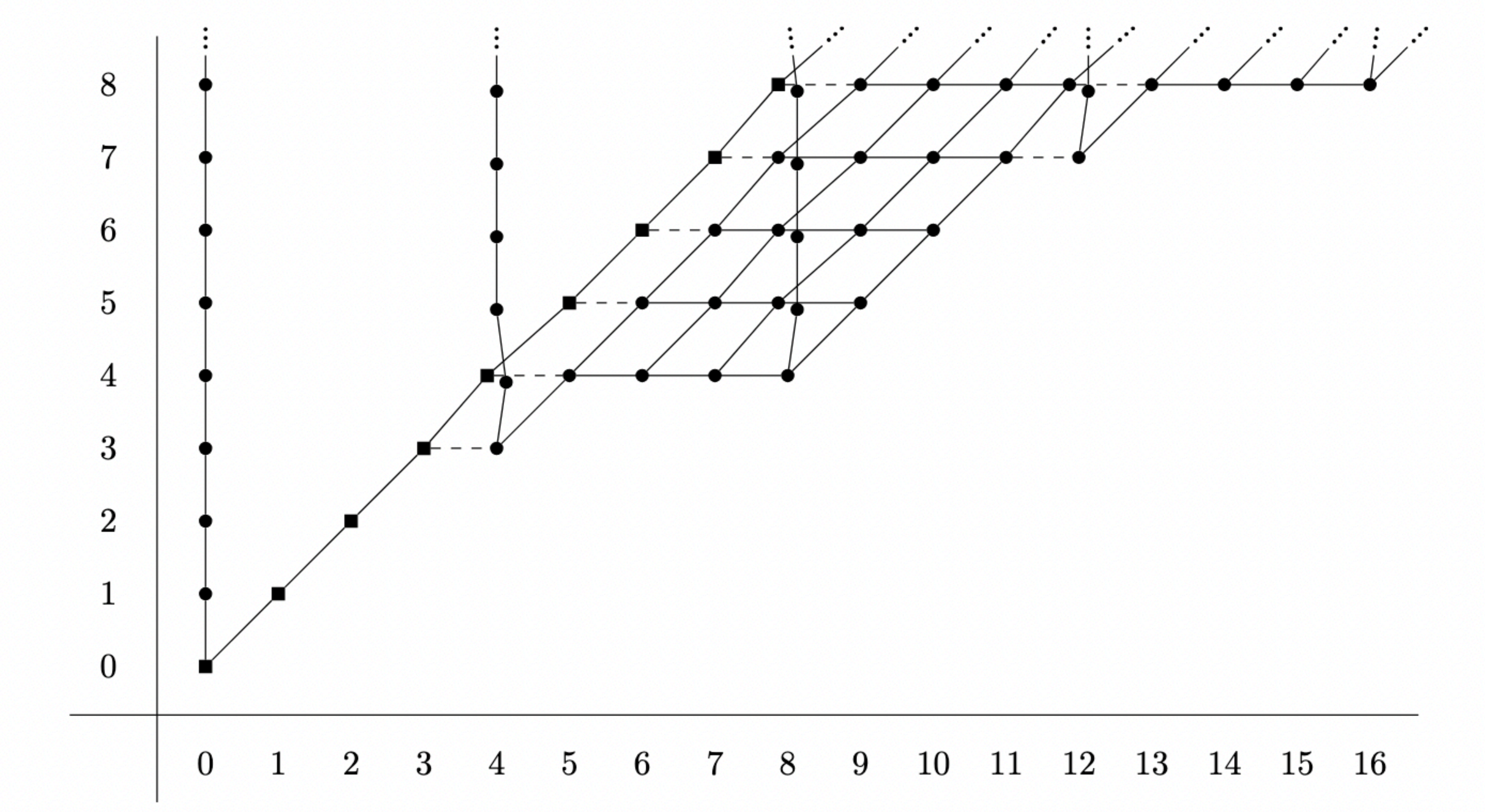

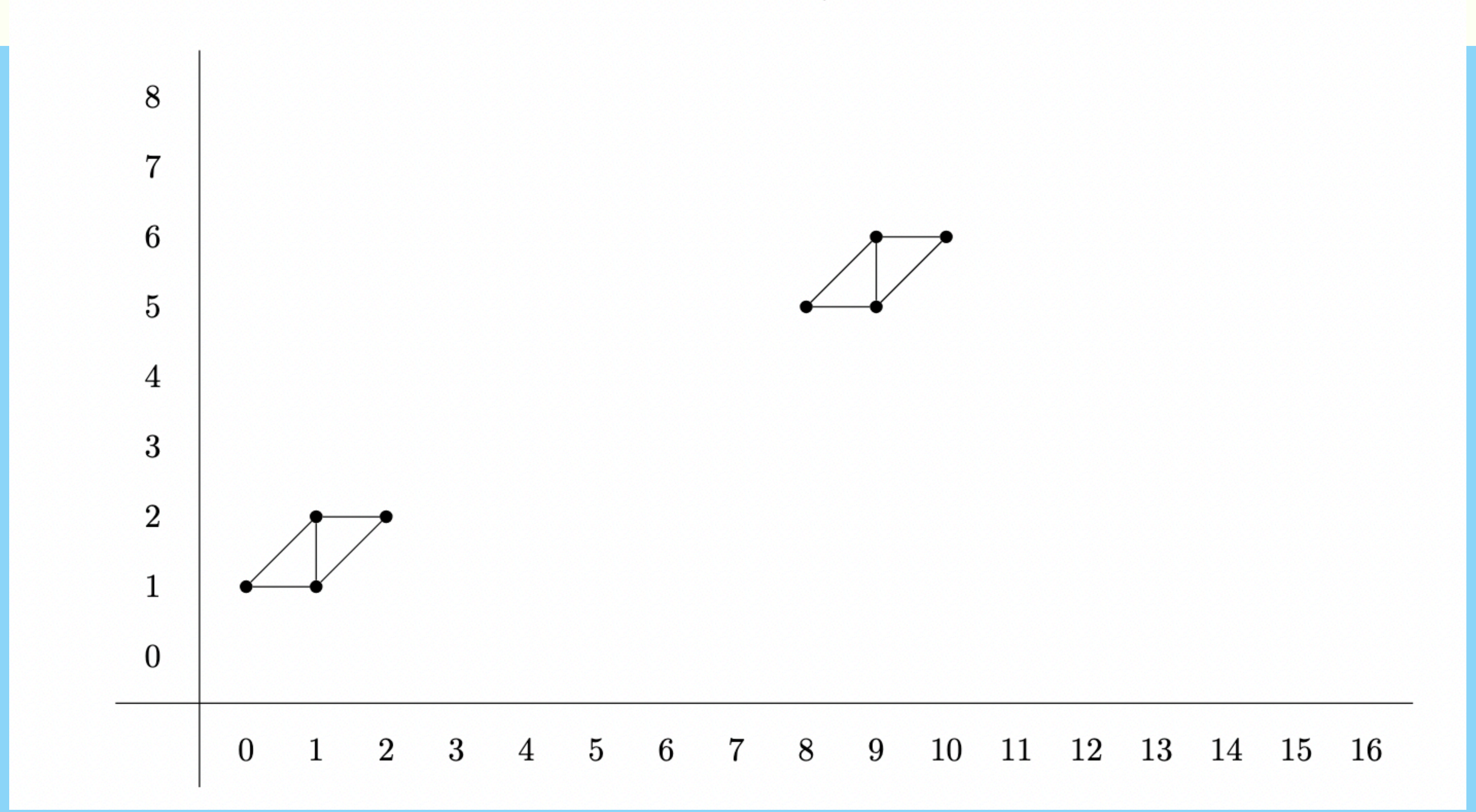

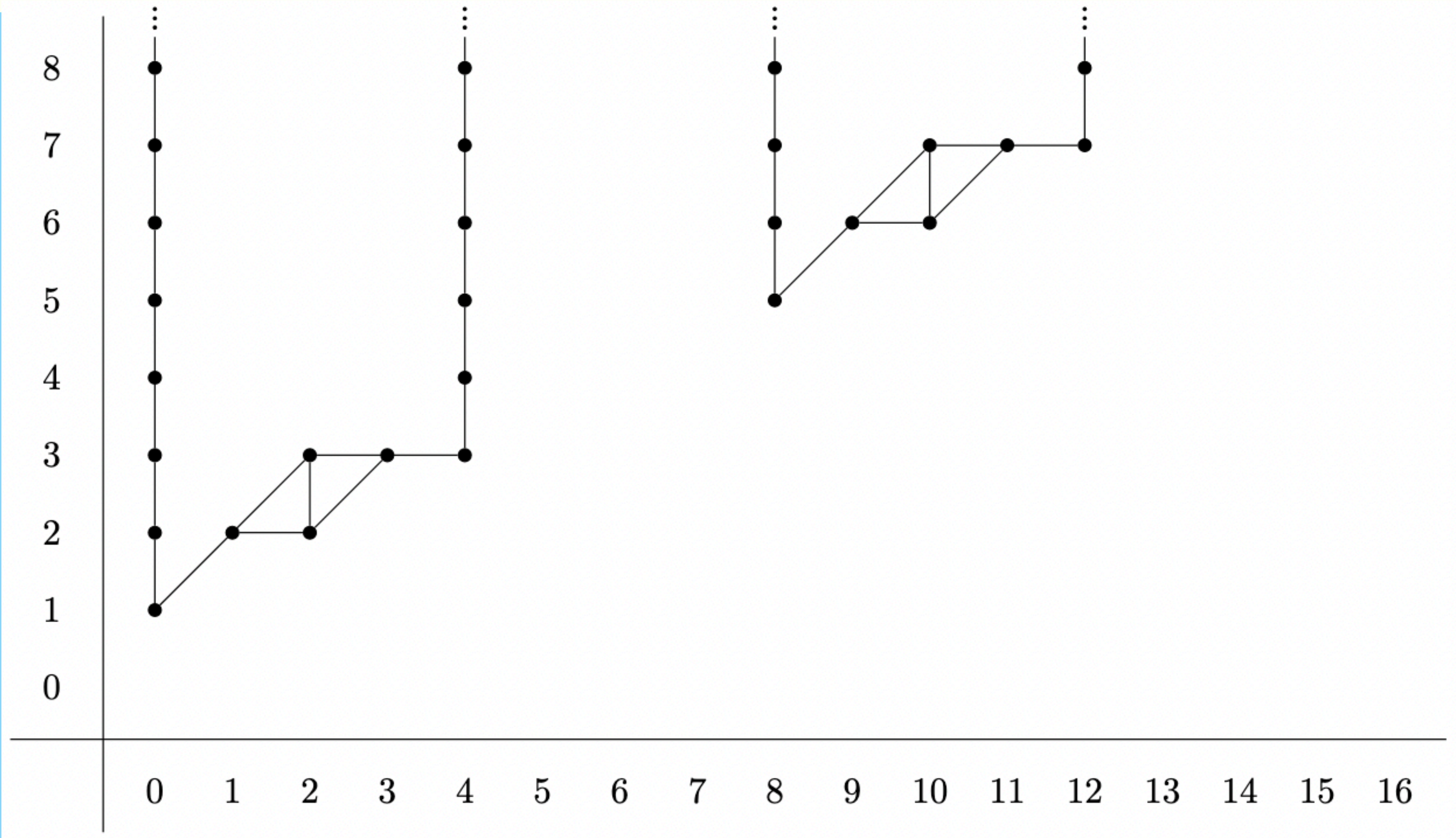

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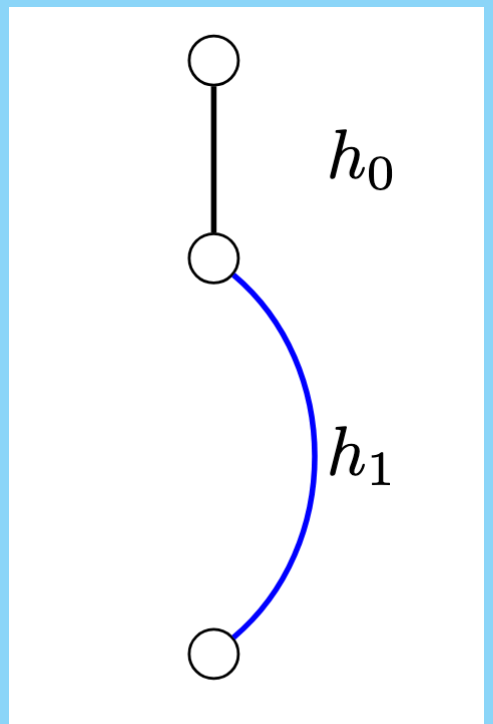

Algebraic Atiyah–Hirzebruch spectral sequence

$$\bigoplus_{\text{cells of } B_0^F(1)} \mathsf{Ext}^{***}_{A(1)}(\mathbb{M}_2, \mathbb{M}_2) \implies \mathsf{Ext}^{***}_{A(1)}(\mathbb{M}_2, B_0^F(1))$$

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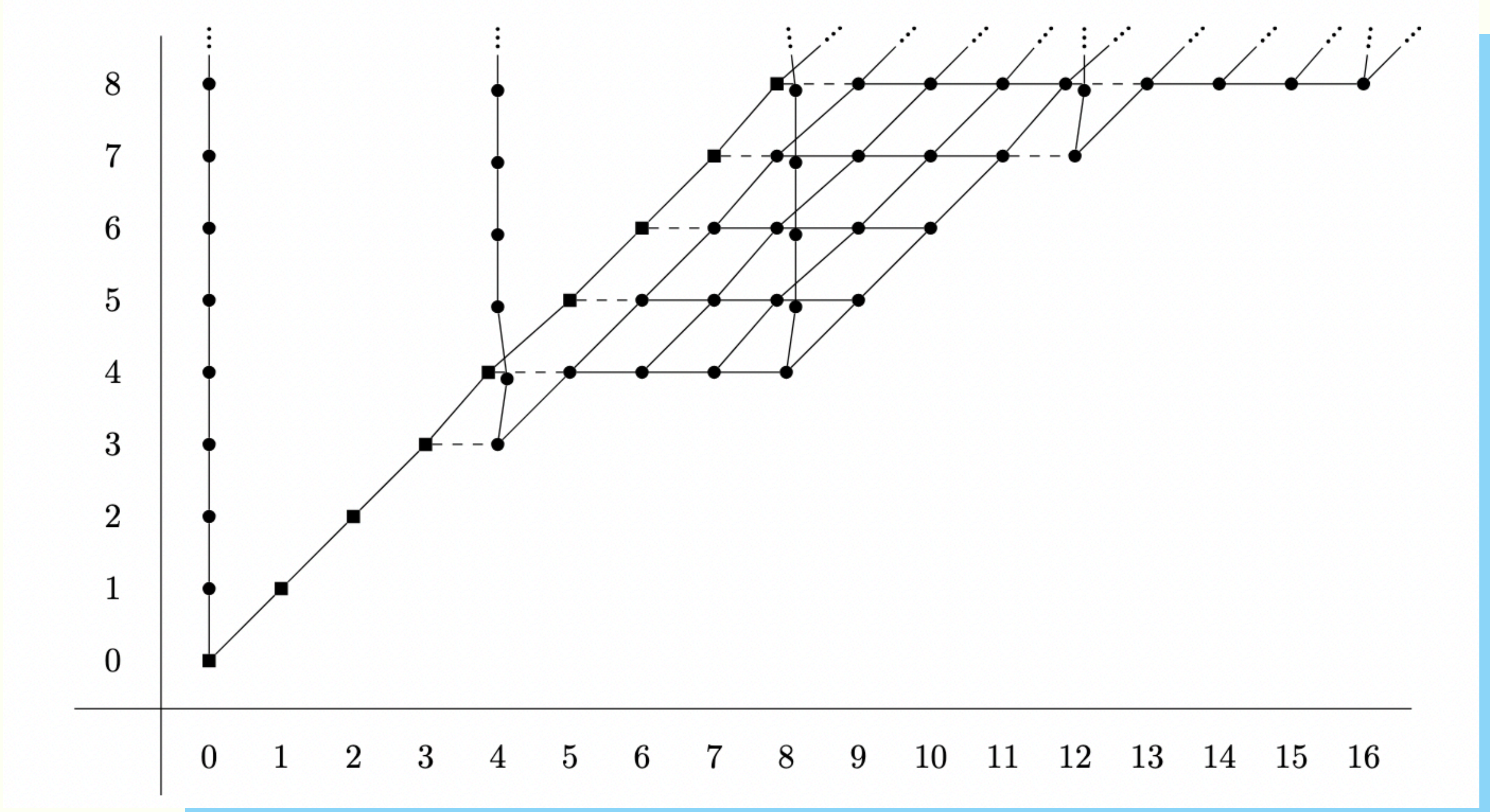

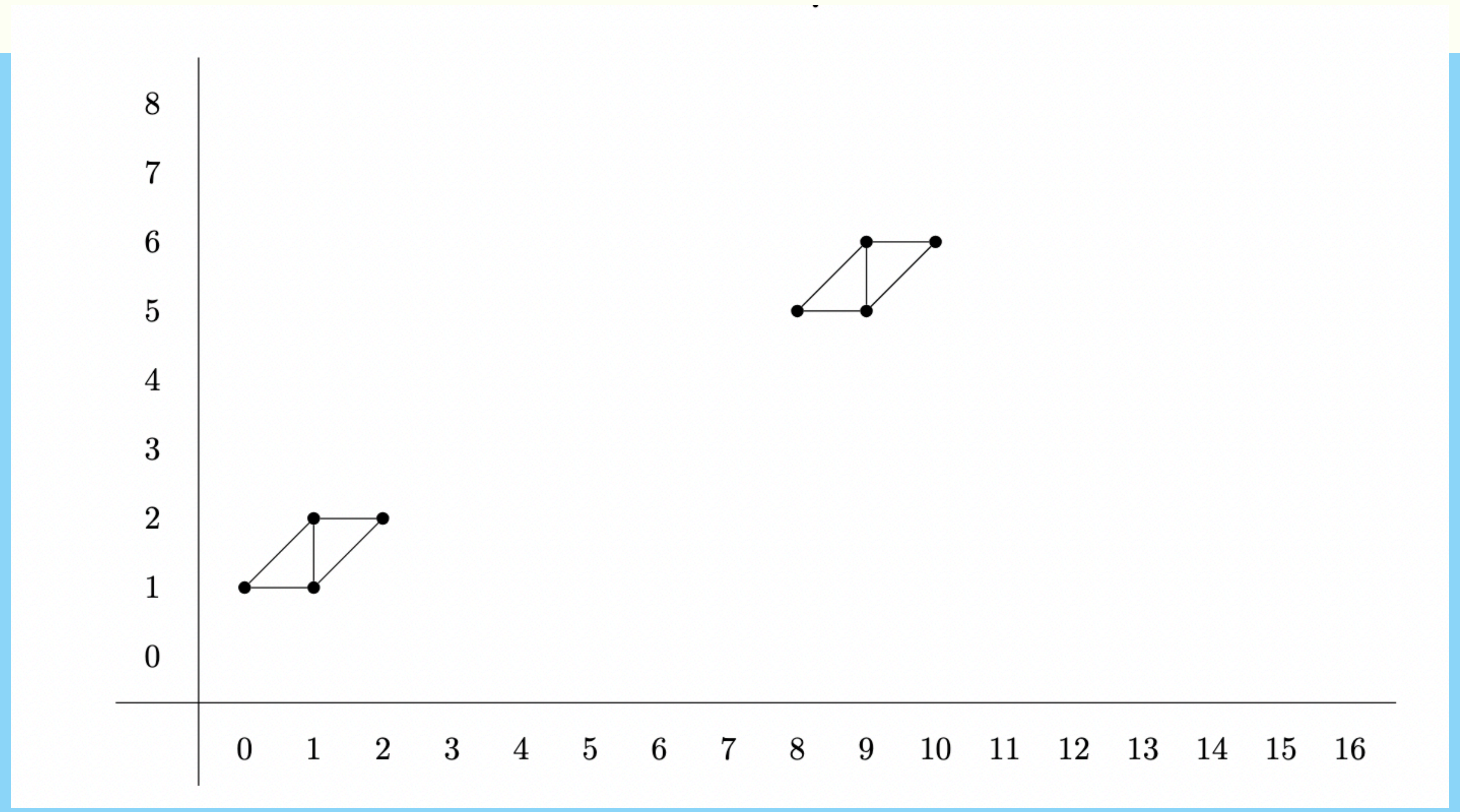

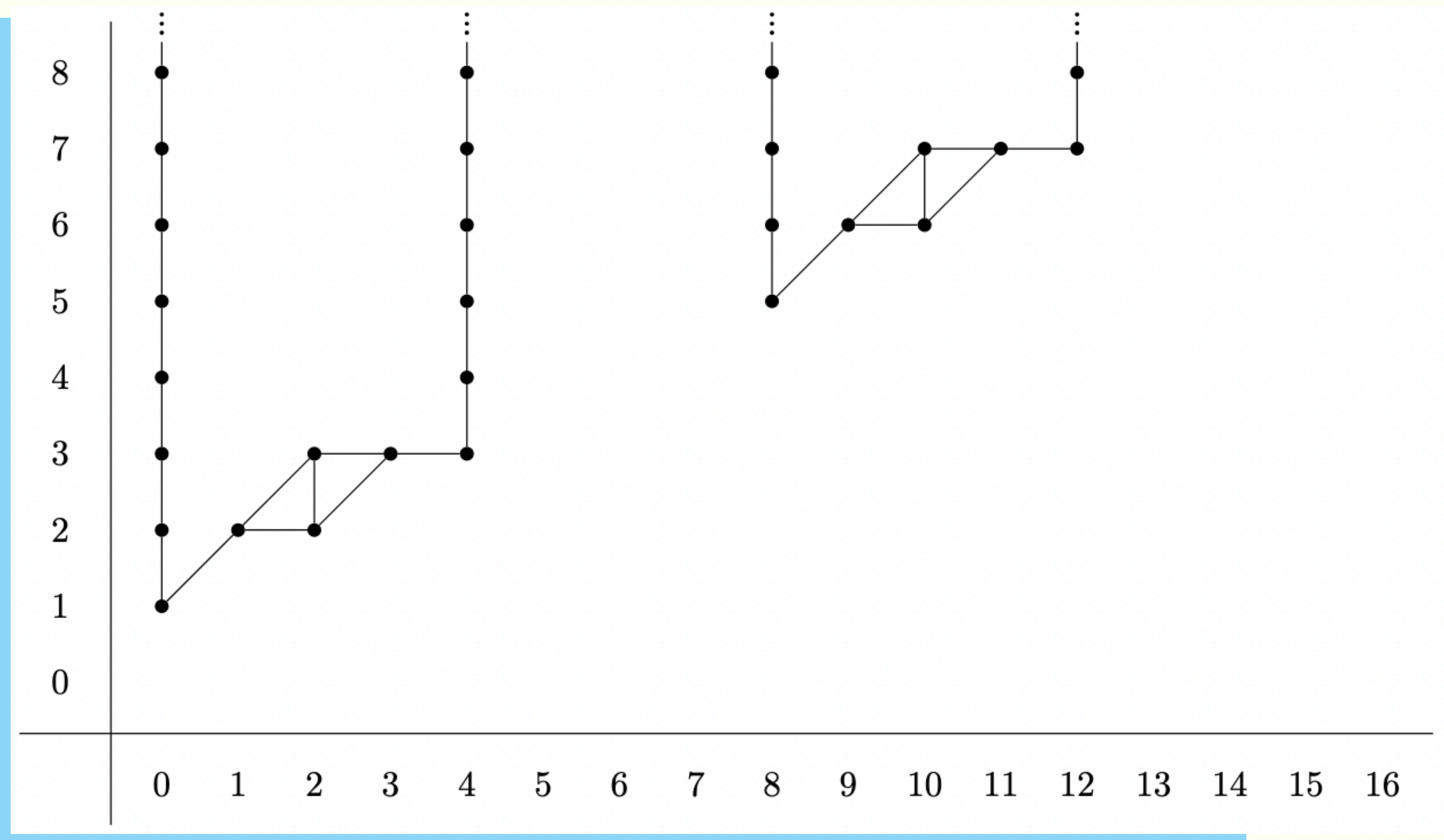

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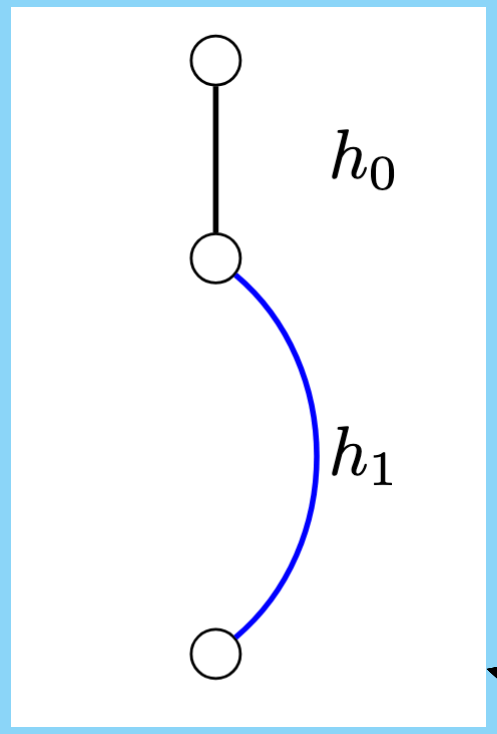
$B_0^F(1) \cong$ 

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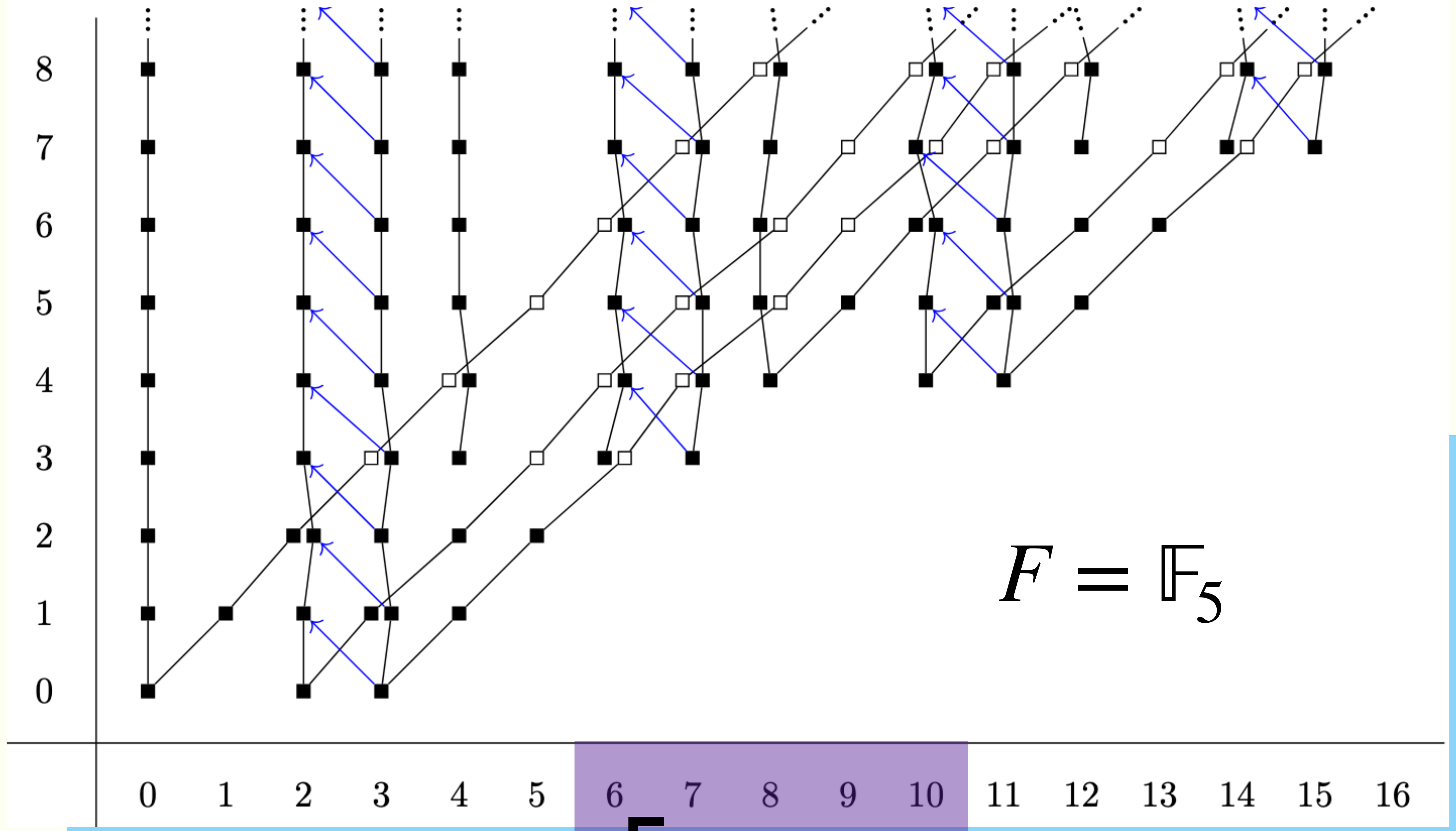
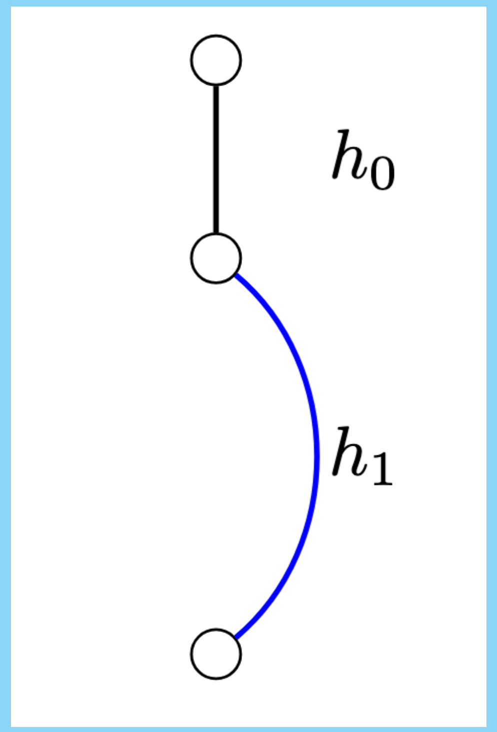
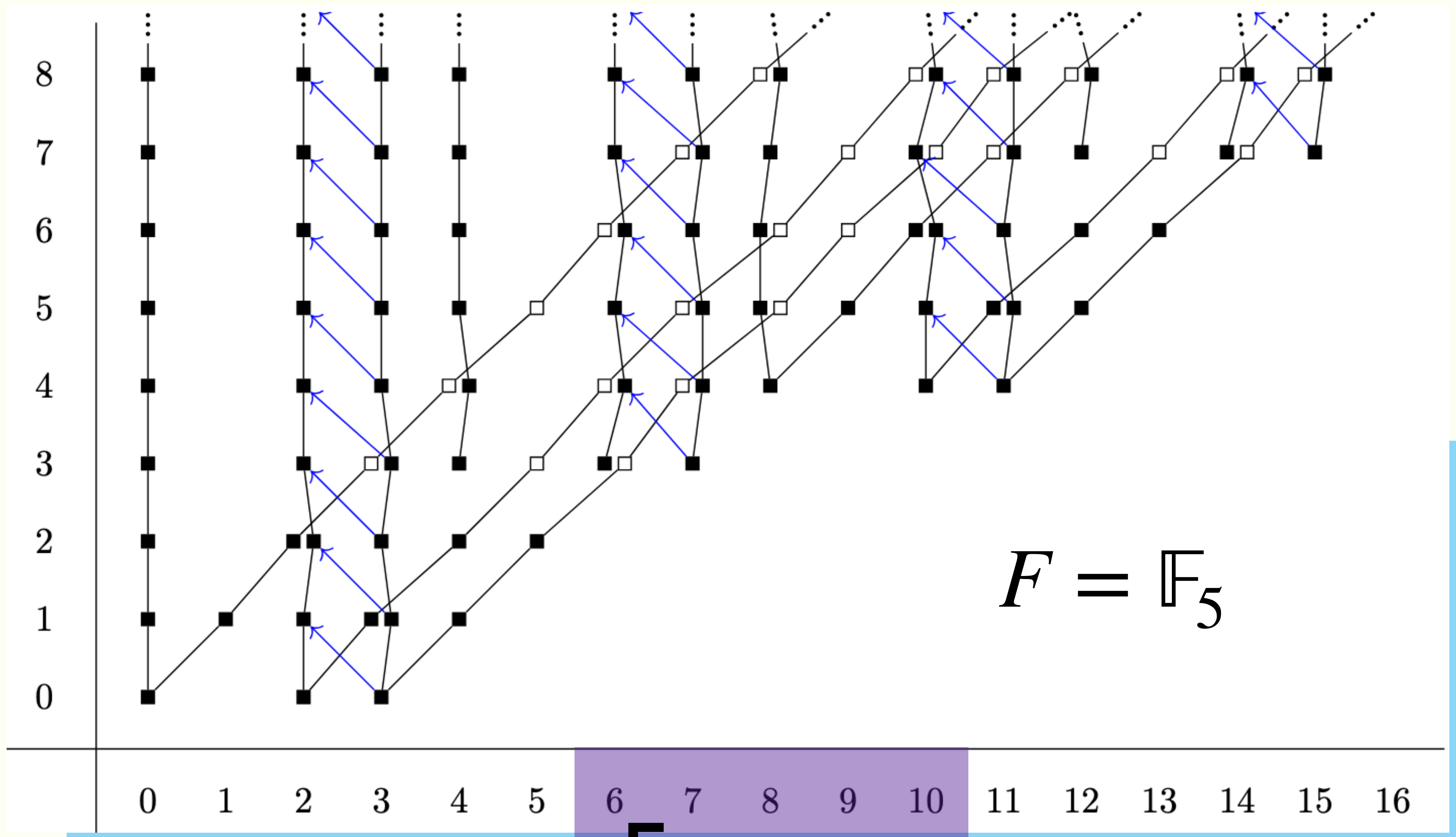
Differentials given by attaching maps



E$_1$-page

$B_0^F(1) \cong$ 



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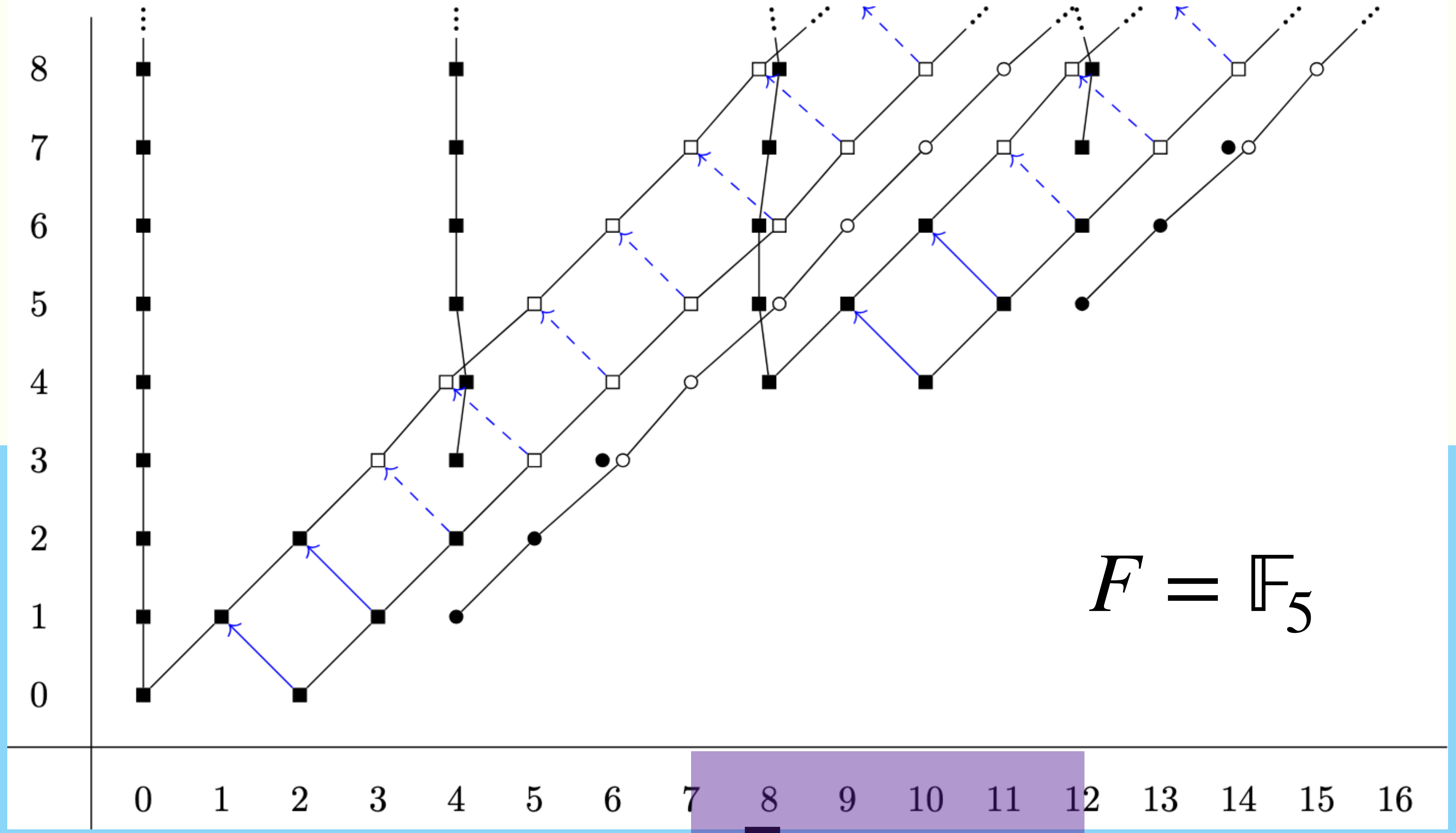
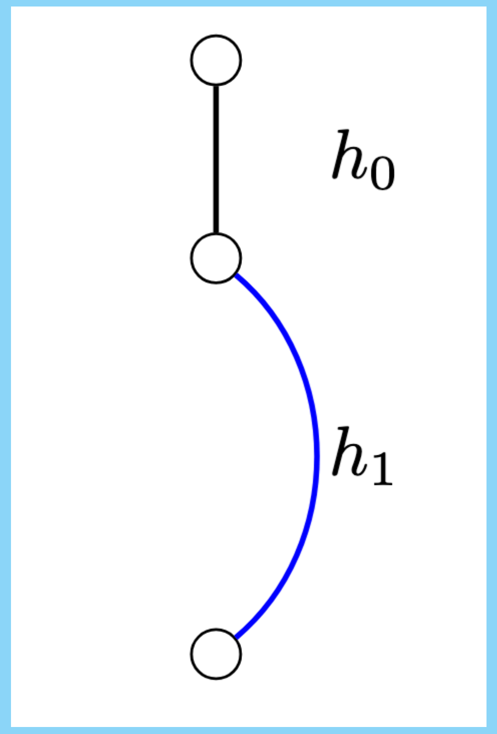
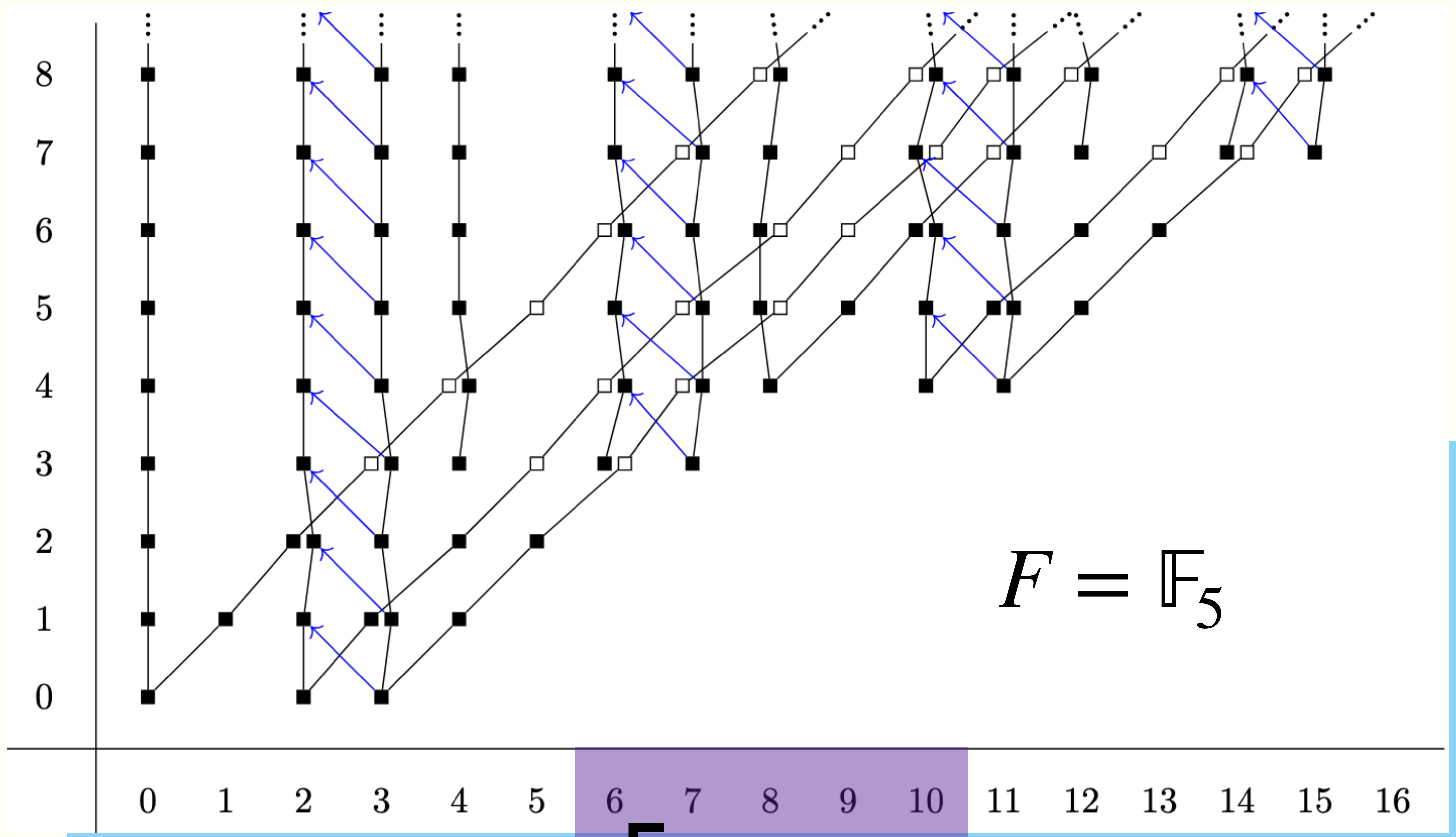


E_2-page

$B_0^F(1) \cong$ 



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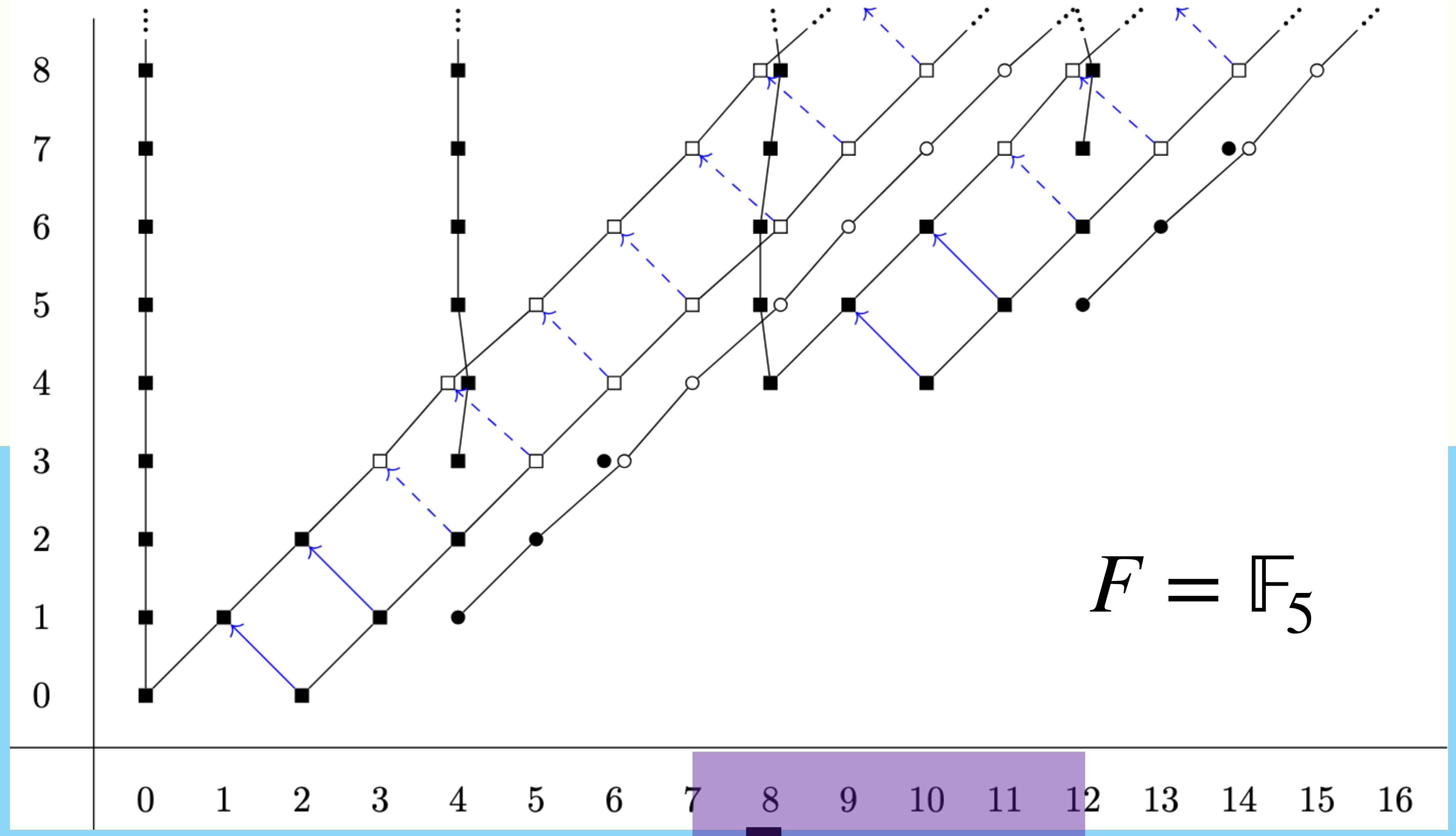

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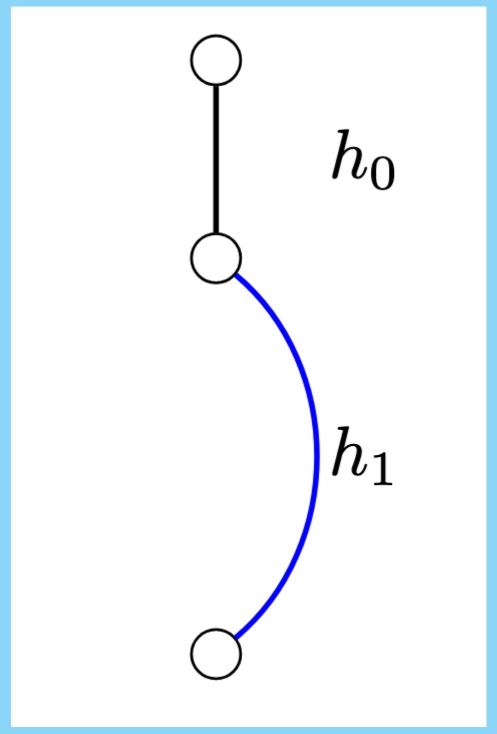
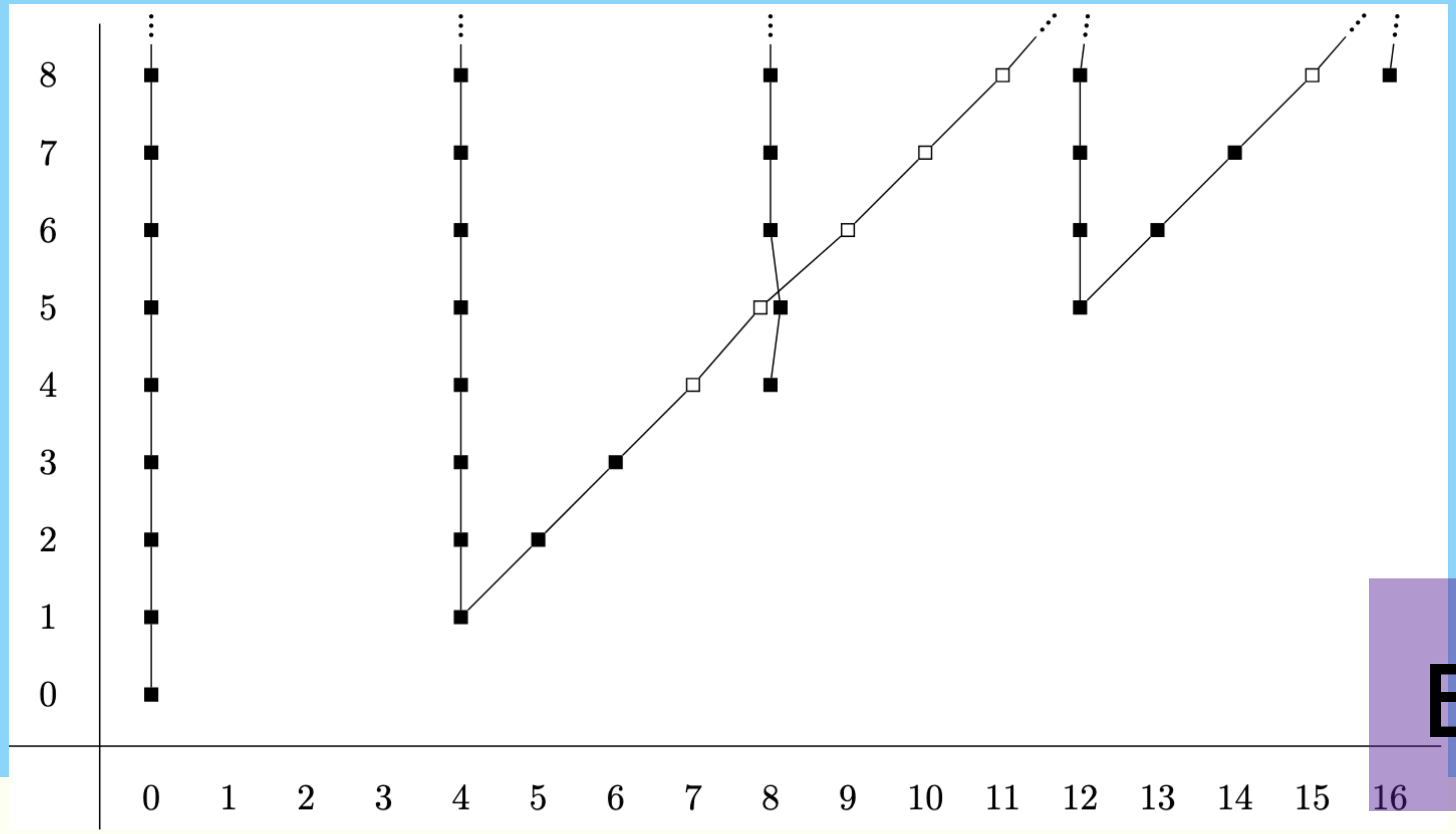
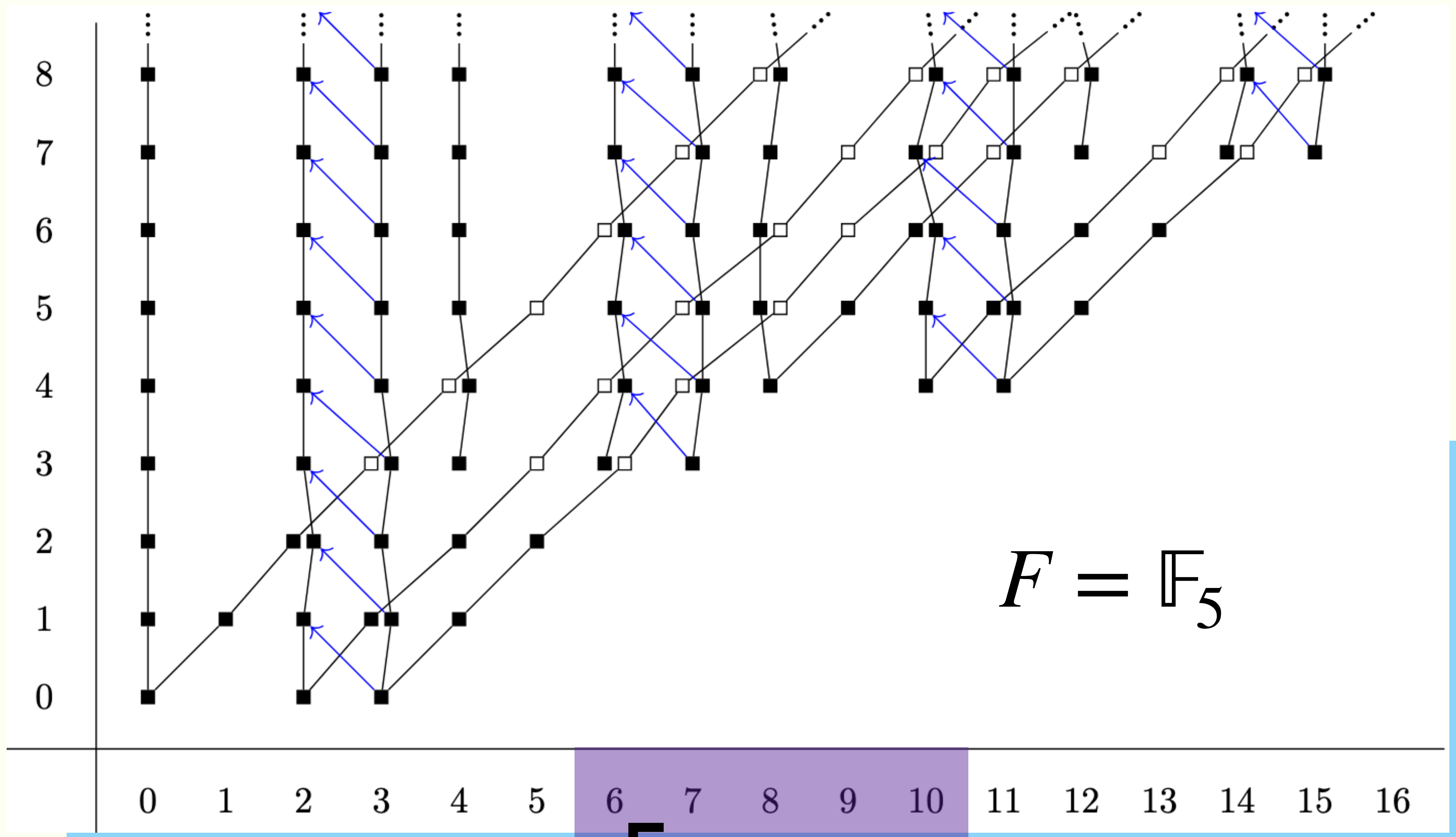




$E_3 = E_\infty$-page



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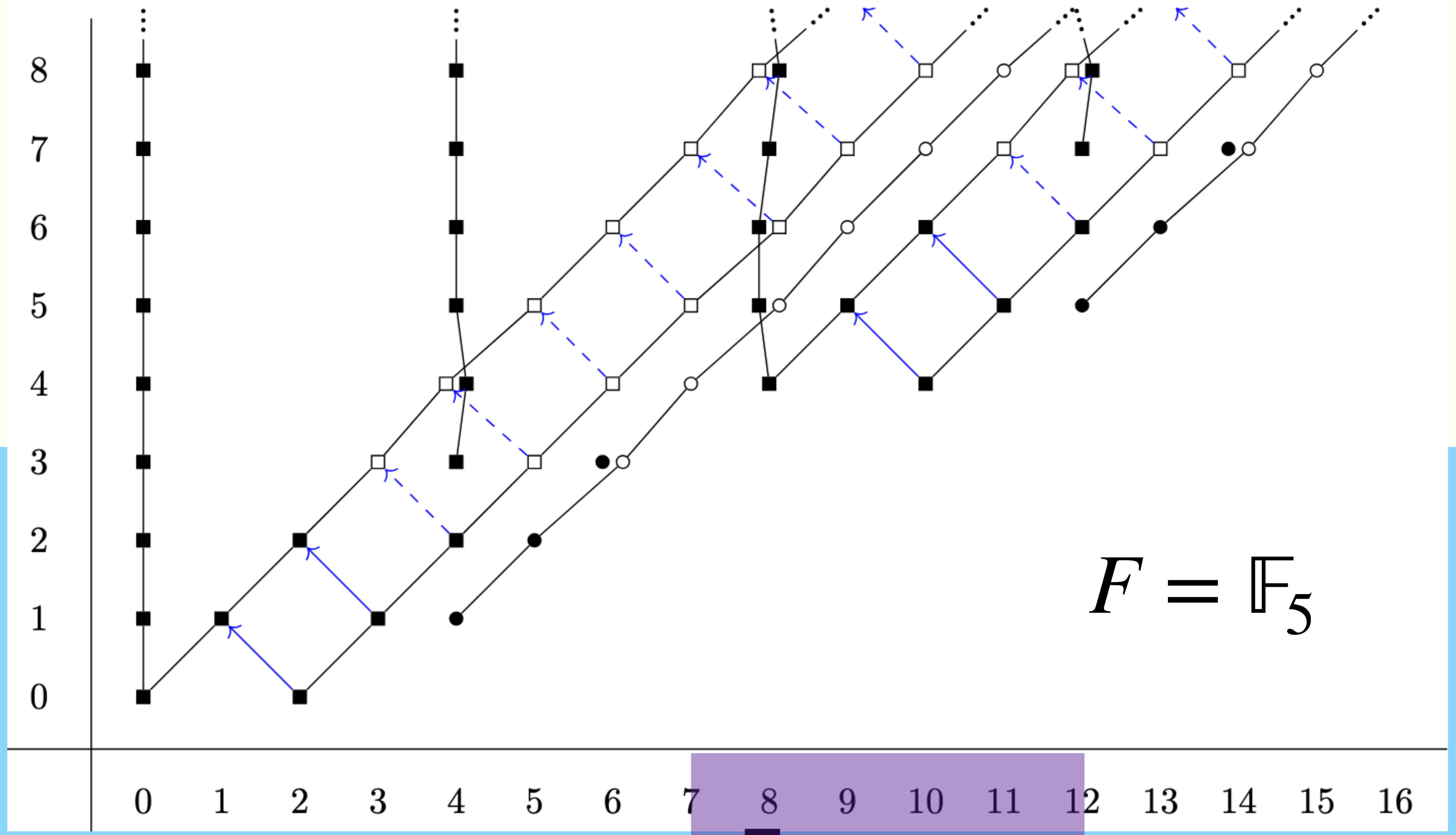

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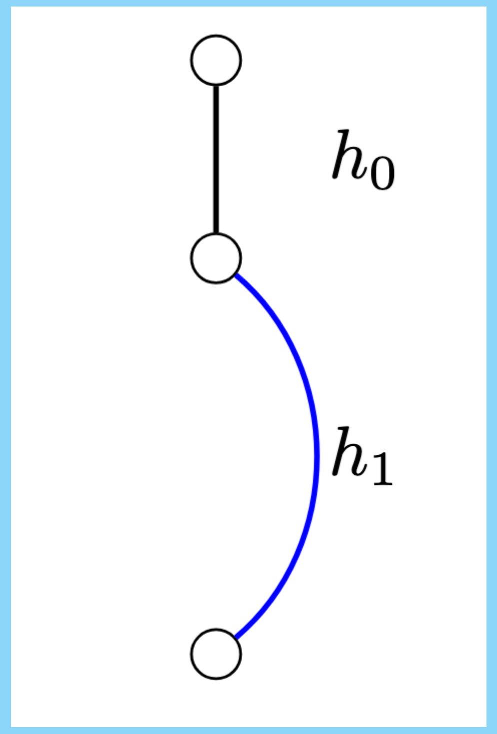
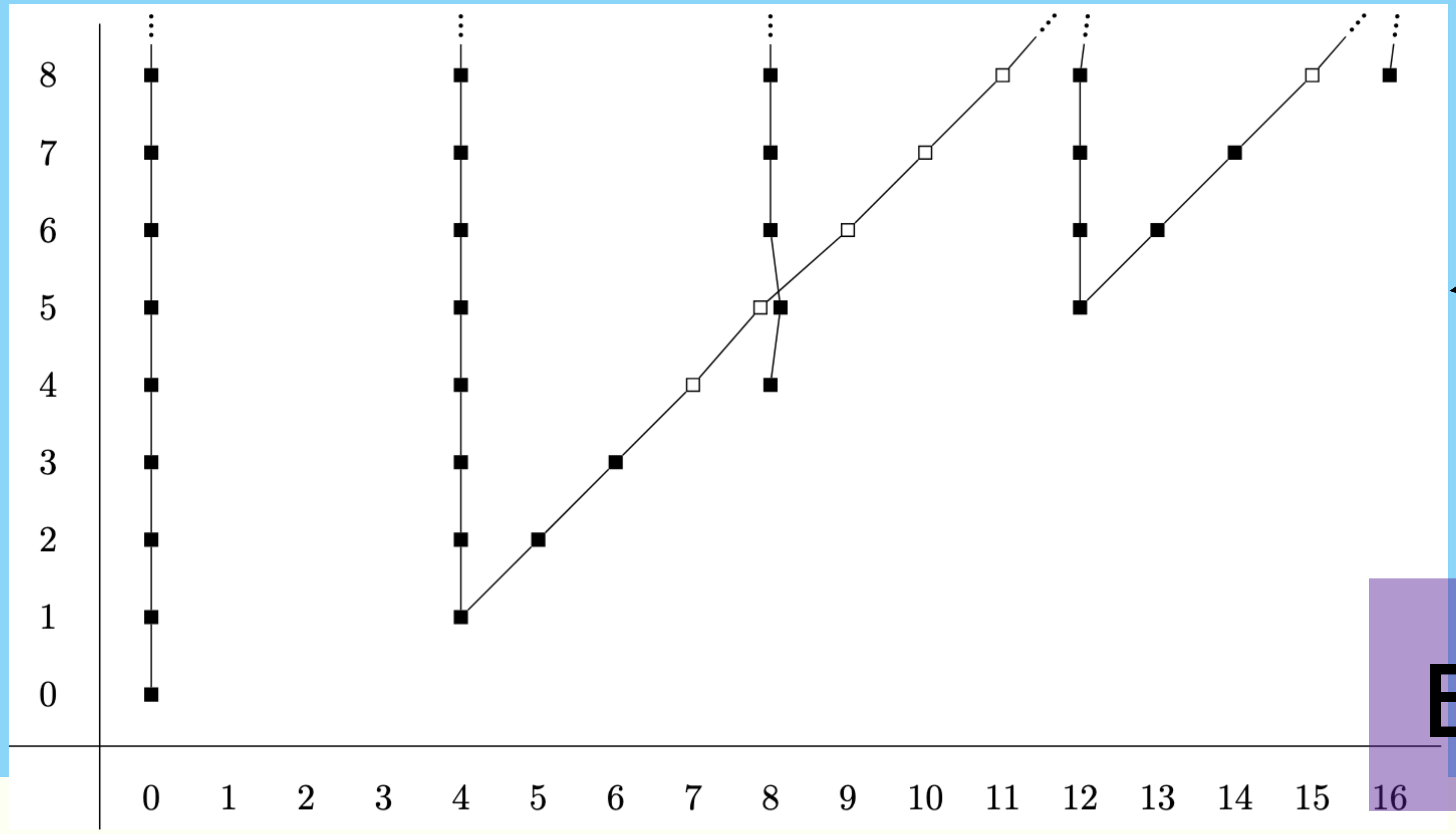
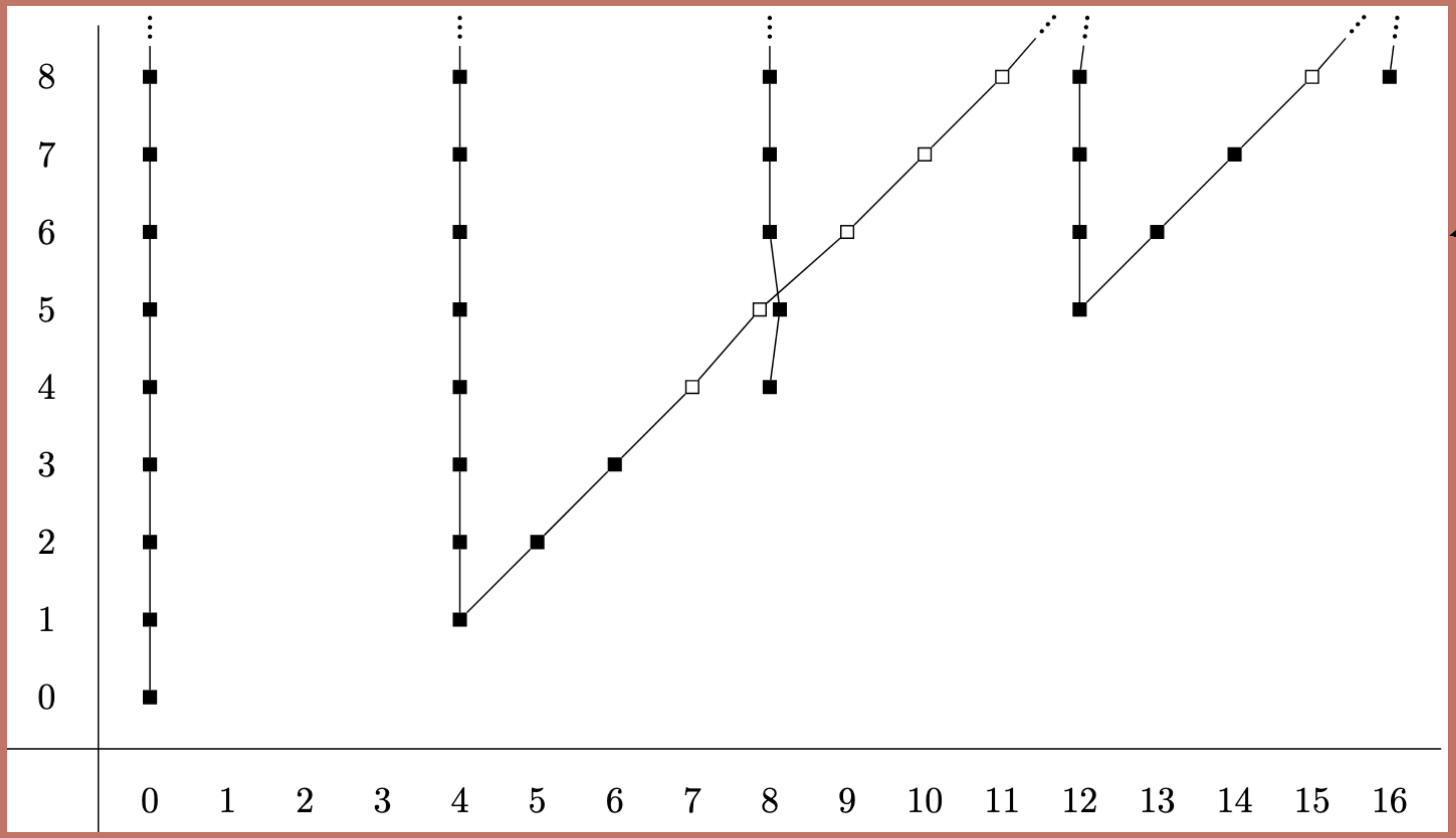




This is $\mathrm{Ext}_{A(1)}^{***}(\mathbb{M}_2, B_0^F(1))$

$E_3 = E_\infty$-page



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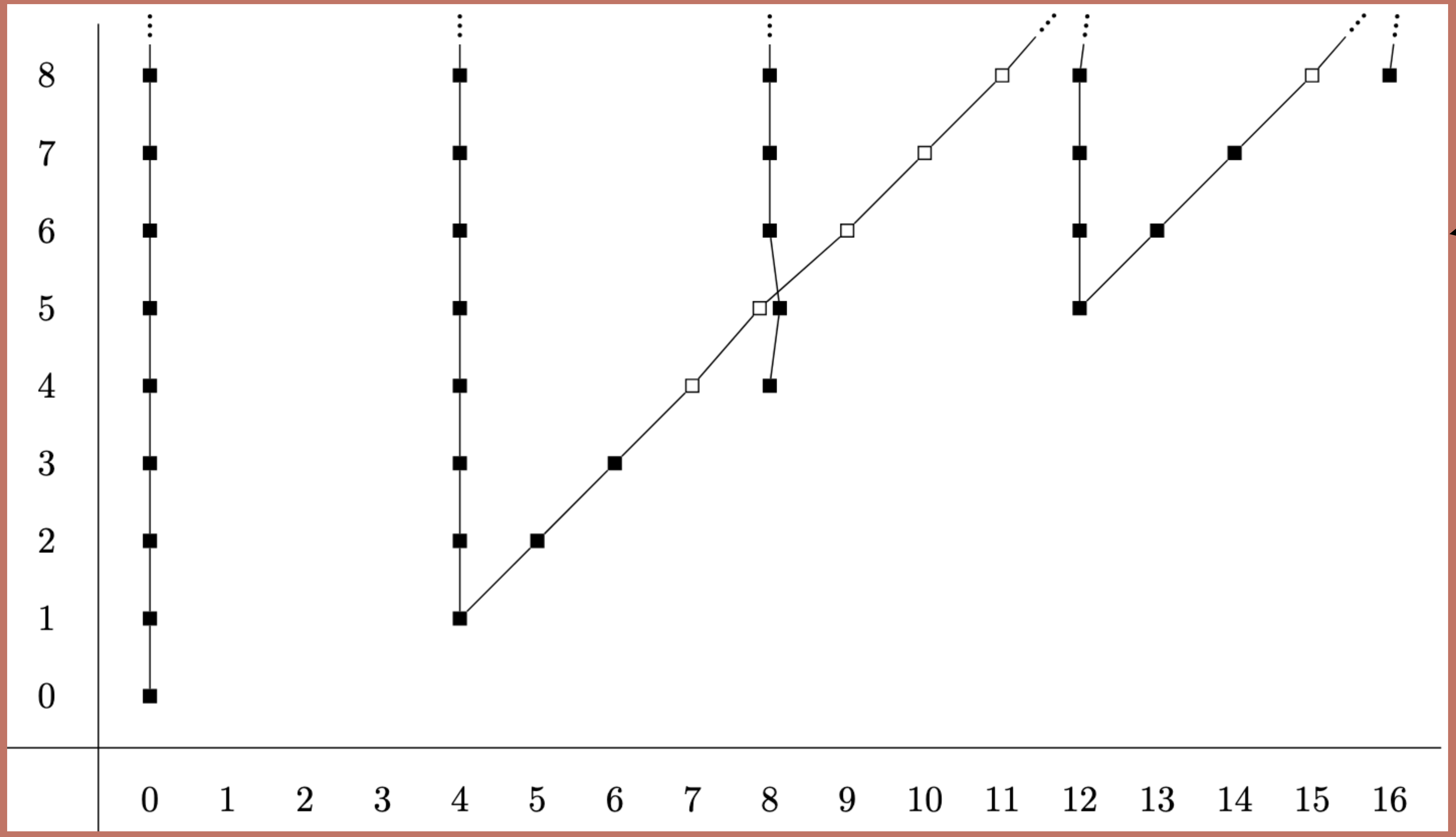
Over $\mathbb{R}$, there are 9 images to keep track of...and a d_3!

This was only for $B_0^F(1)$...
do we have to do it again?



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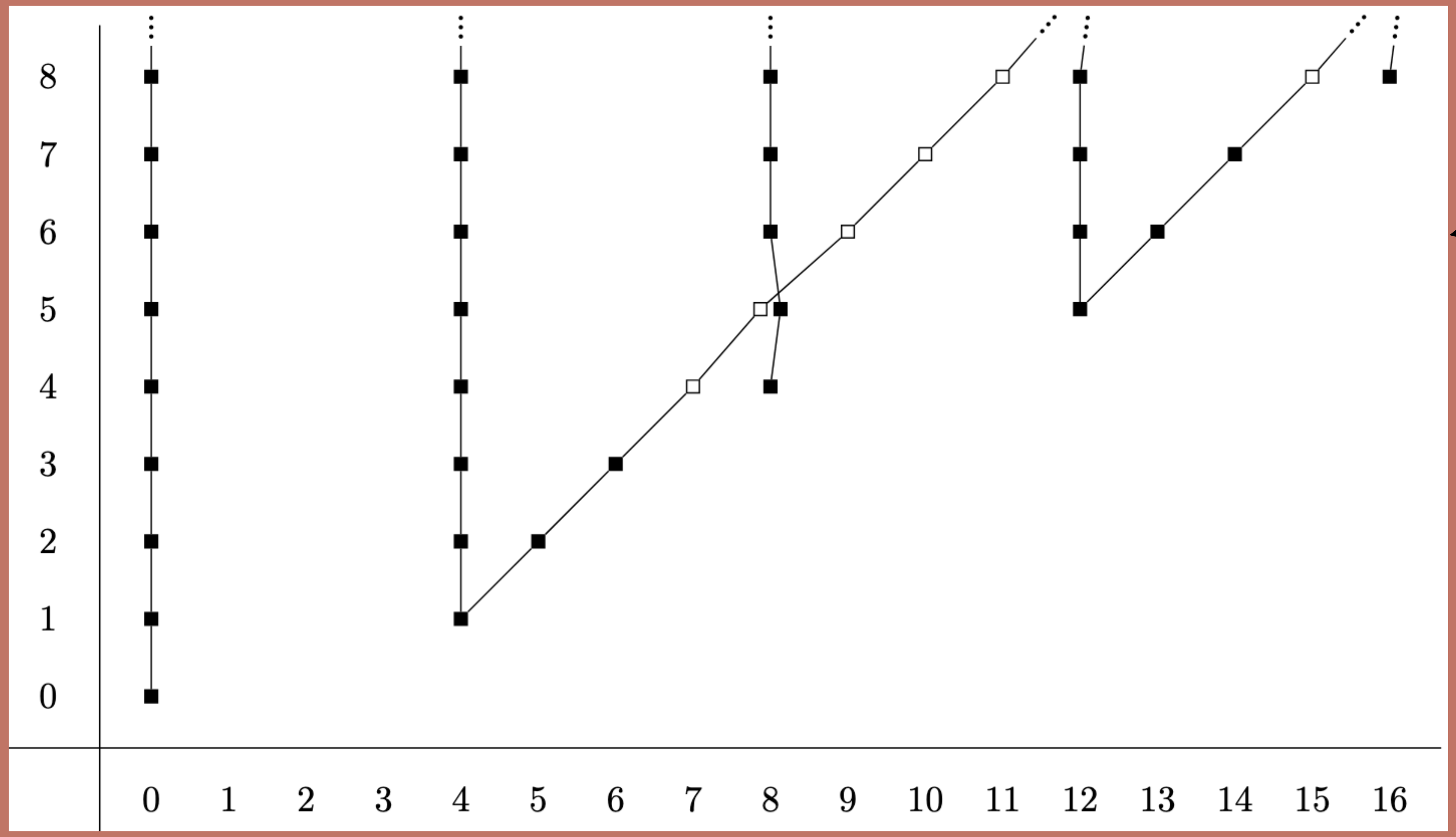

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Short exact sequences

$$0 \to \Sigma^{4k,2k} B_0^F(k) \to B_0^F(2k) \to B_1^F(k-1) \otimes A(1)//A(0)^\vee \to 0$$

NOPE!

$$0 \to \Sigma^{4k,2k} B_0^F(k) \otimes B_0^F(1) \to B_0^F(2k+1) \to B_1^F(k-1) \otimes A(1)//A(0) \to 0$$

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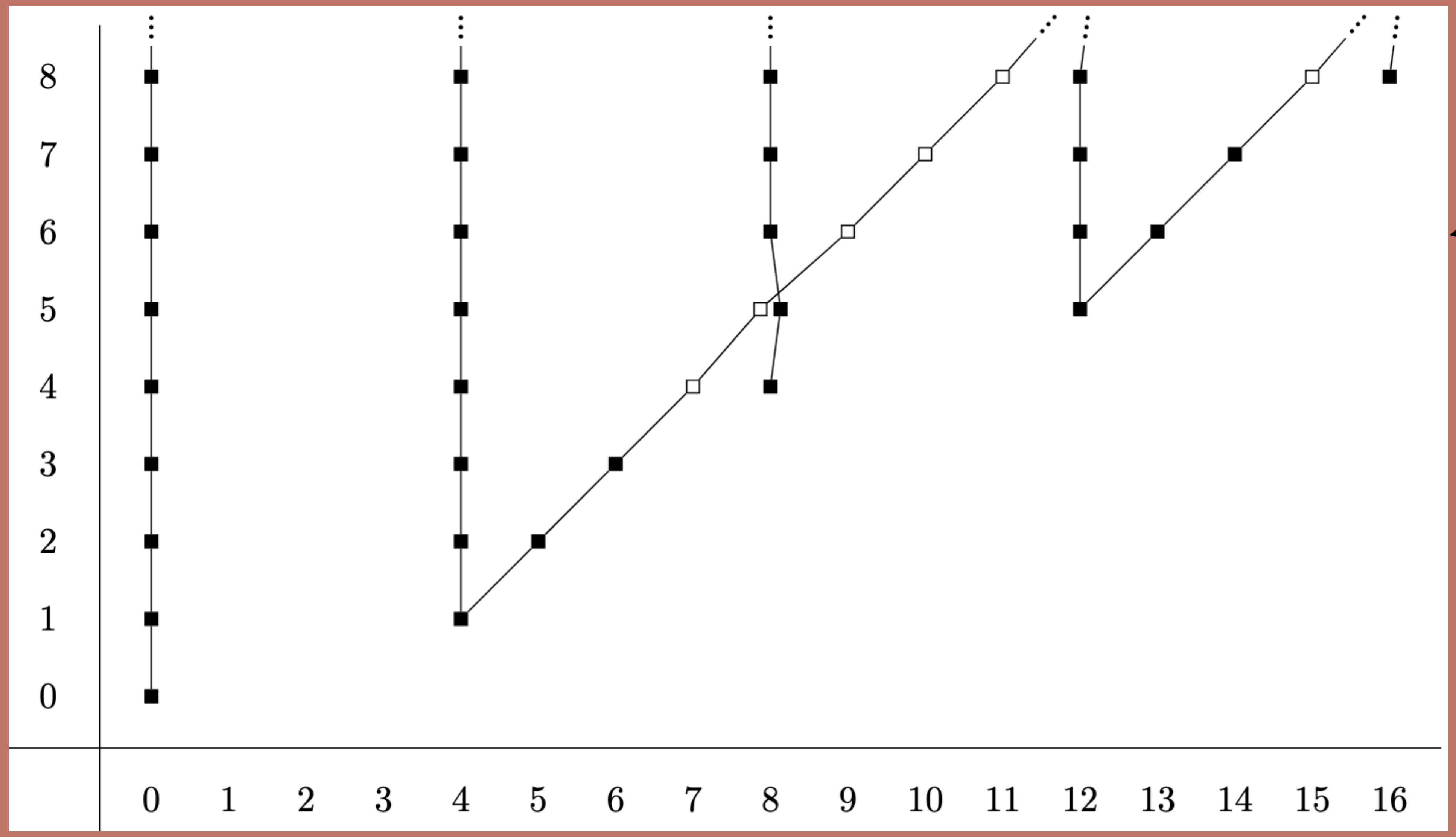

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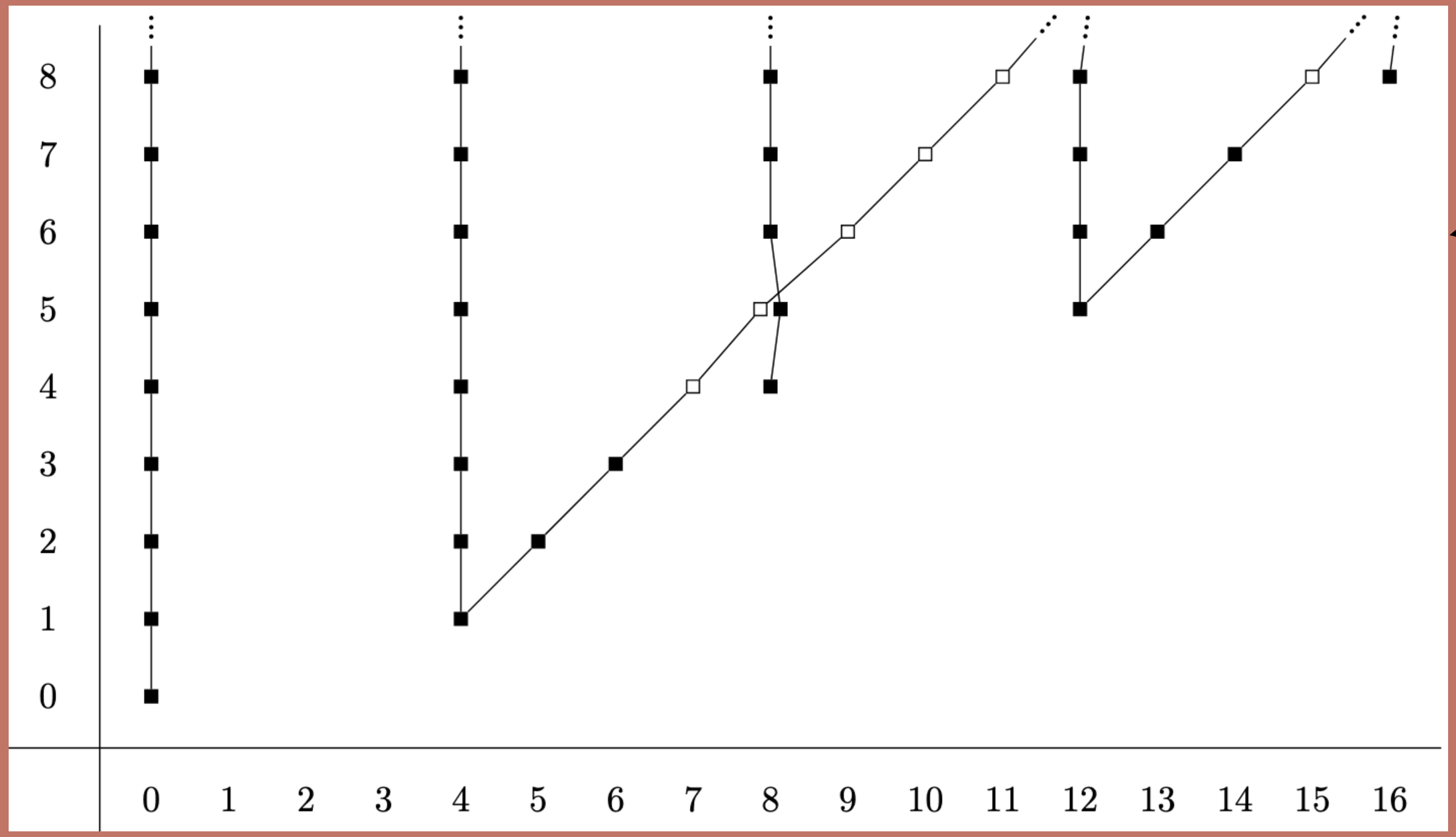
Induction!

Apply $\mathrm{Ext}^{***}_{A(1)}(\mathbb{M}_2, -)$ to get long exact sequences.

This was only for $B_0^F(1)$...
do we have to do it again?



This is
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Over $\mathbb{R}$, there are 9 images to keep track of...and a d_3!

Short exact sequences

NOPE!

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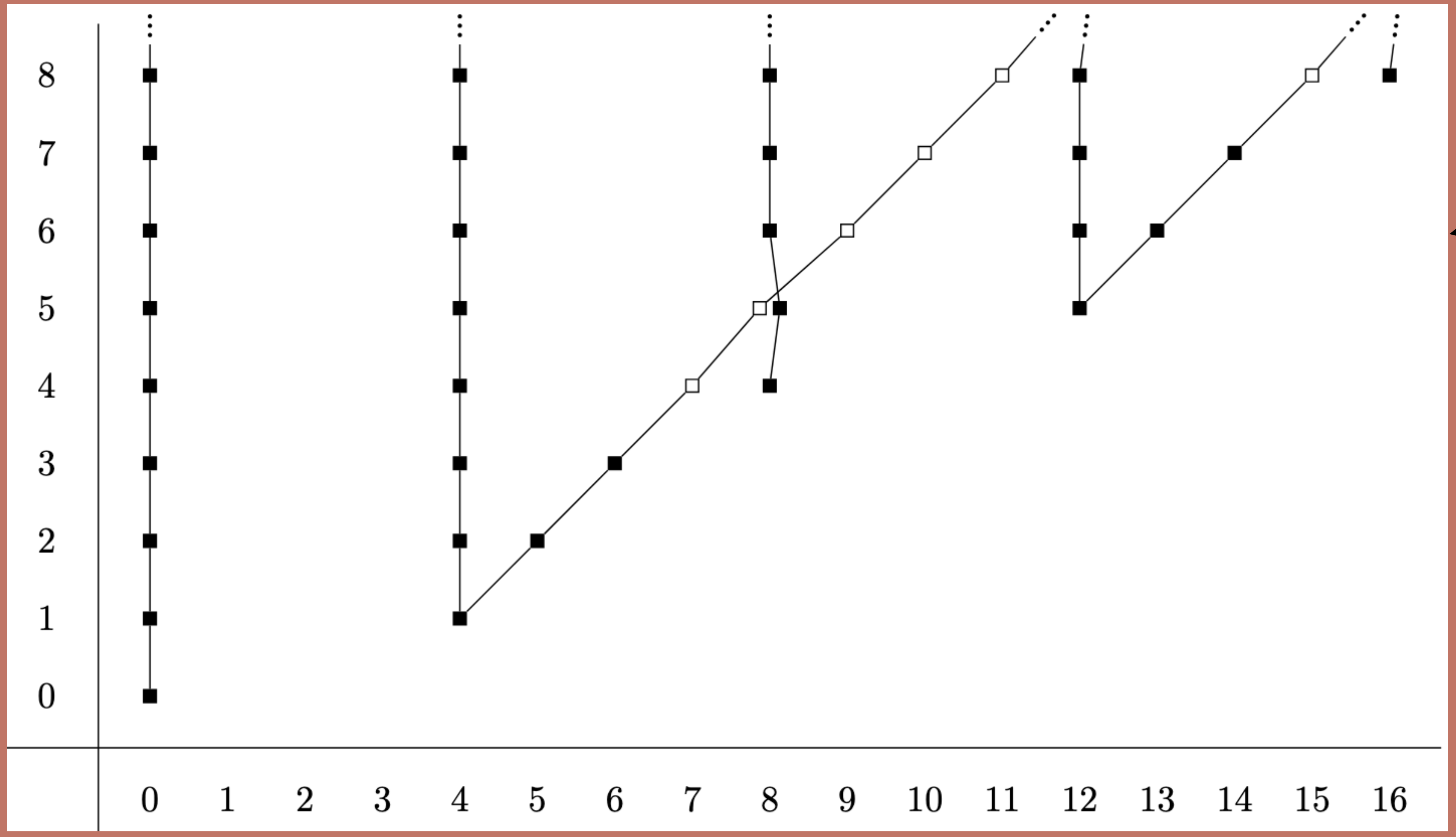
Induction!

Apply $\mathrm{Ext}_{A(1)}^{***}(\mathbb{M}_2, -)$ to get long exact sequences.

This was only for $B_0^F(1)$... do we have to do it again?

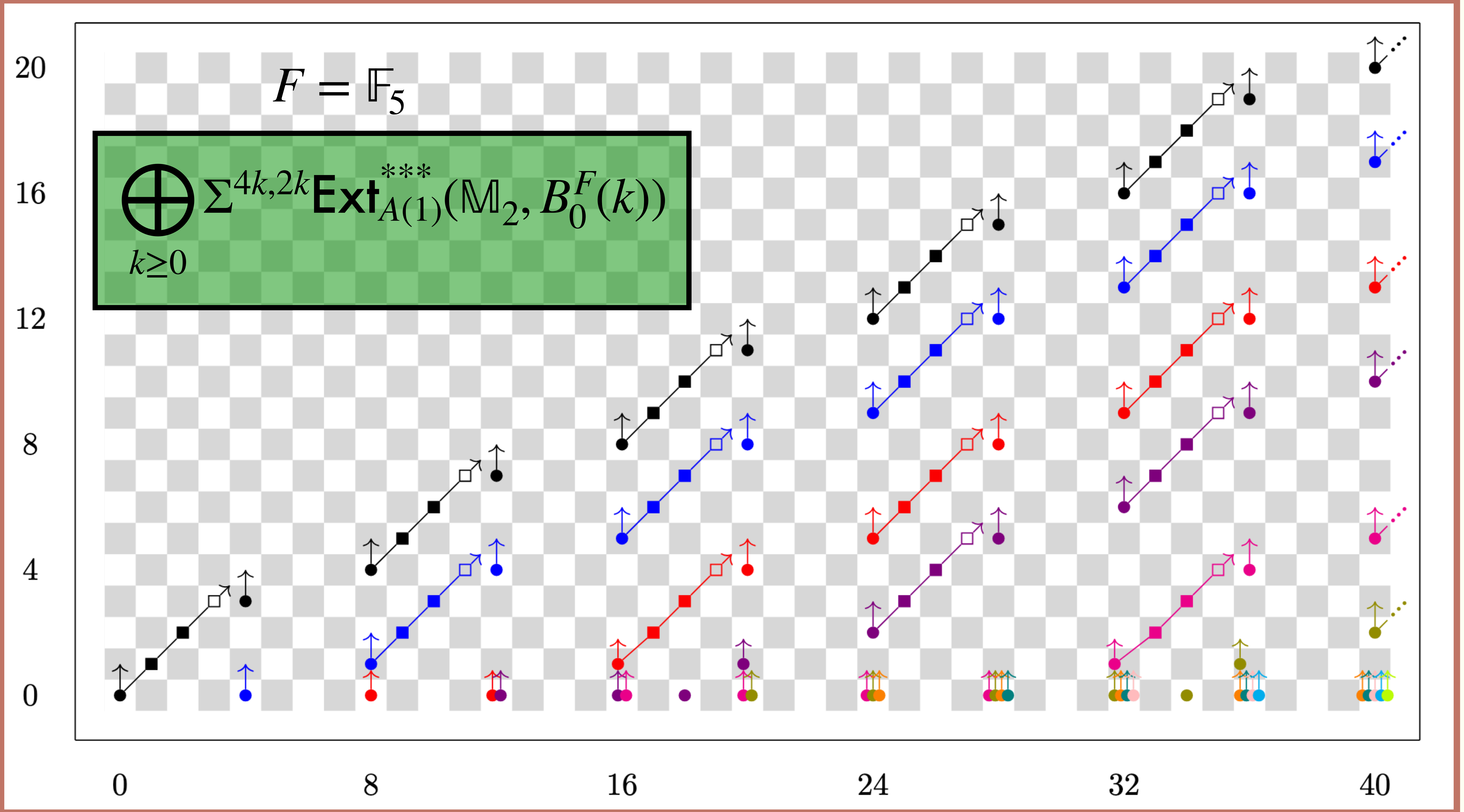
Ok, but what do we get at the end?!



This is $\mathrm{Ext}_{A(1)}^{***}(\mathbb{M}_2, B_0^F(1))$

Over $\mathbb{R}$, there are 9 images to keep track of...and a d_3!

Ok, but what do we get at the end?!



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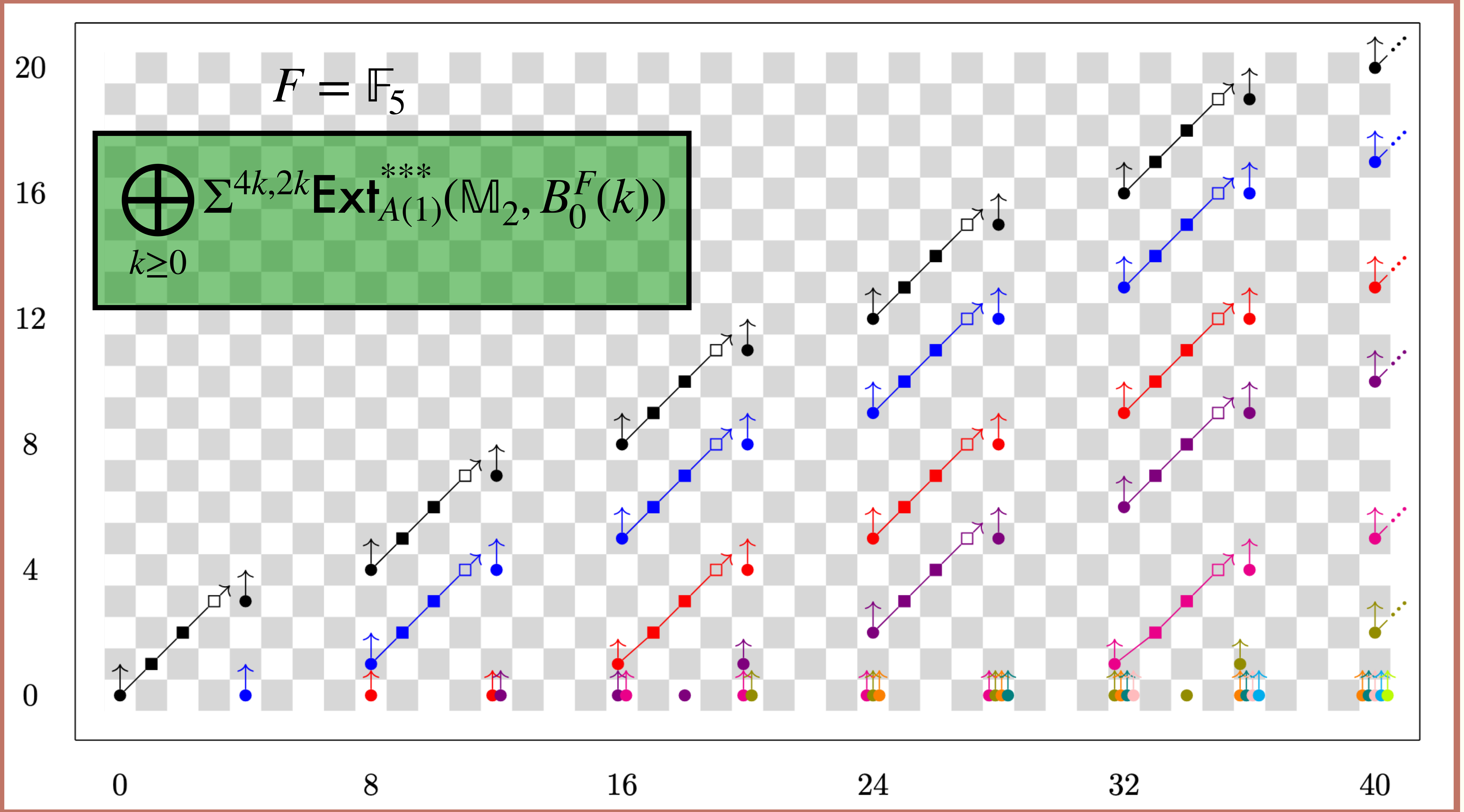

Each color represents a different summand

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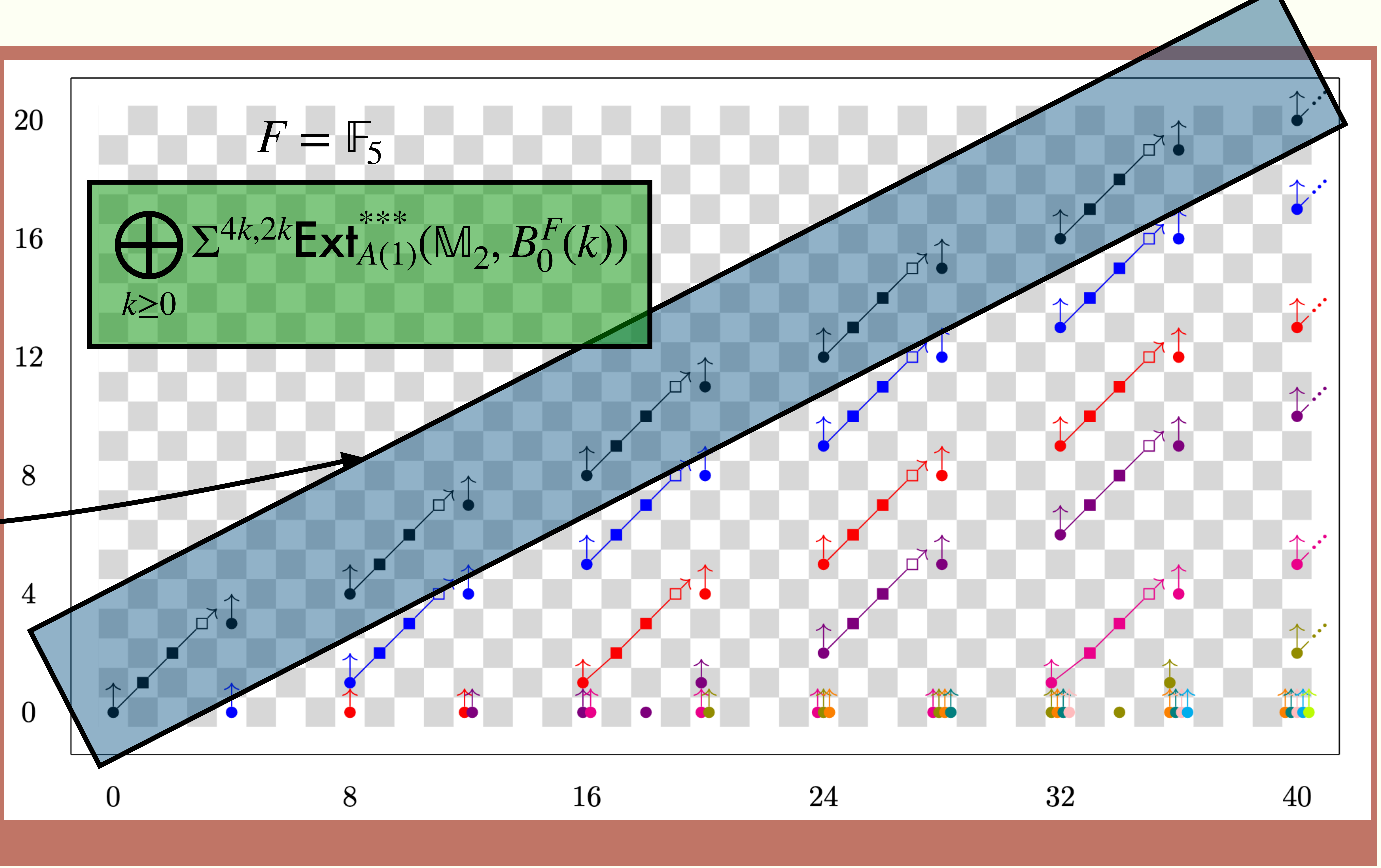
$\mathrm{Ext}^{***}_{A(1)}(\mathbb{M}_2, \mathbb{M}_2)$



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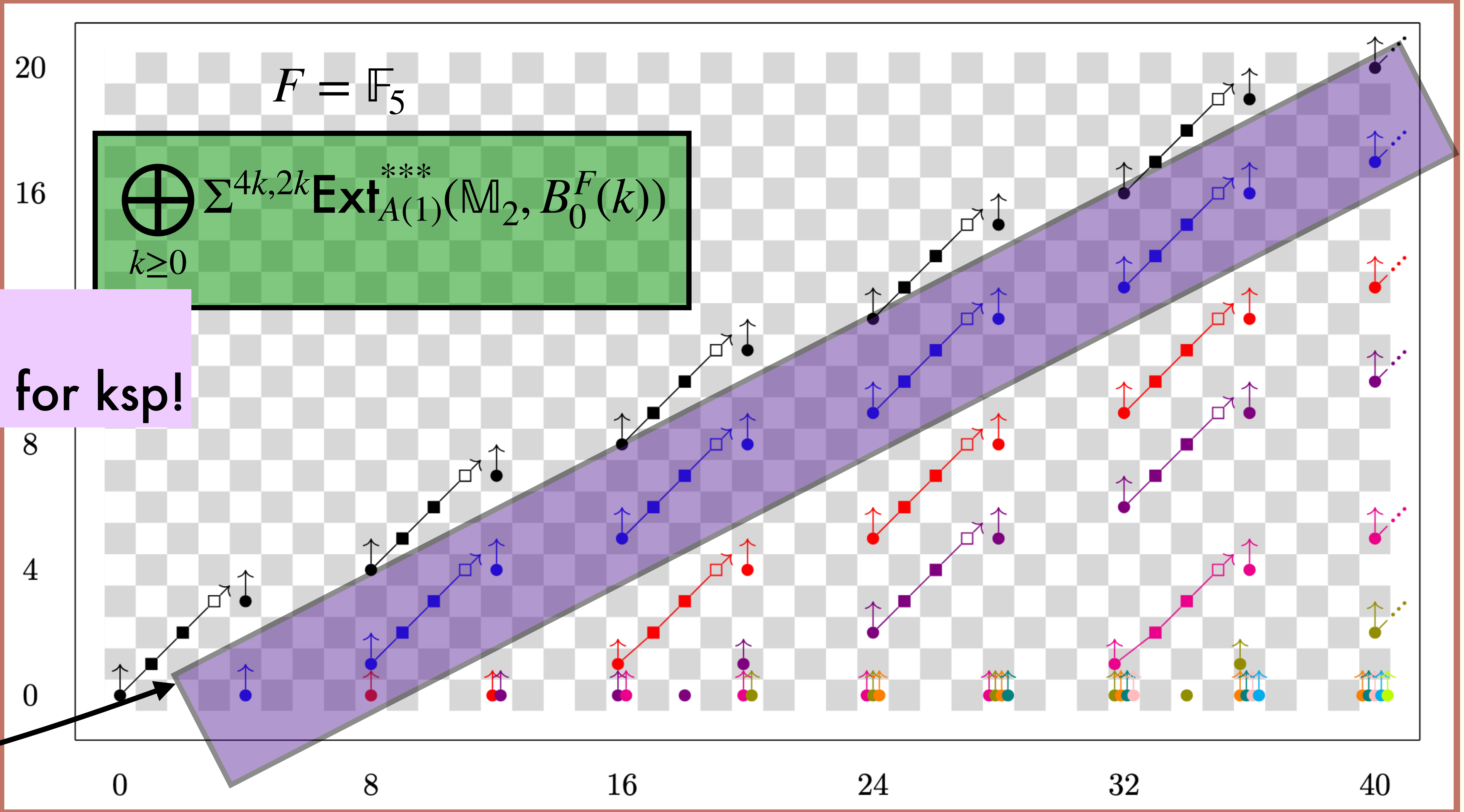
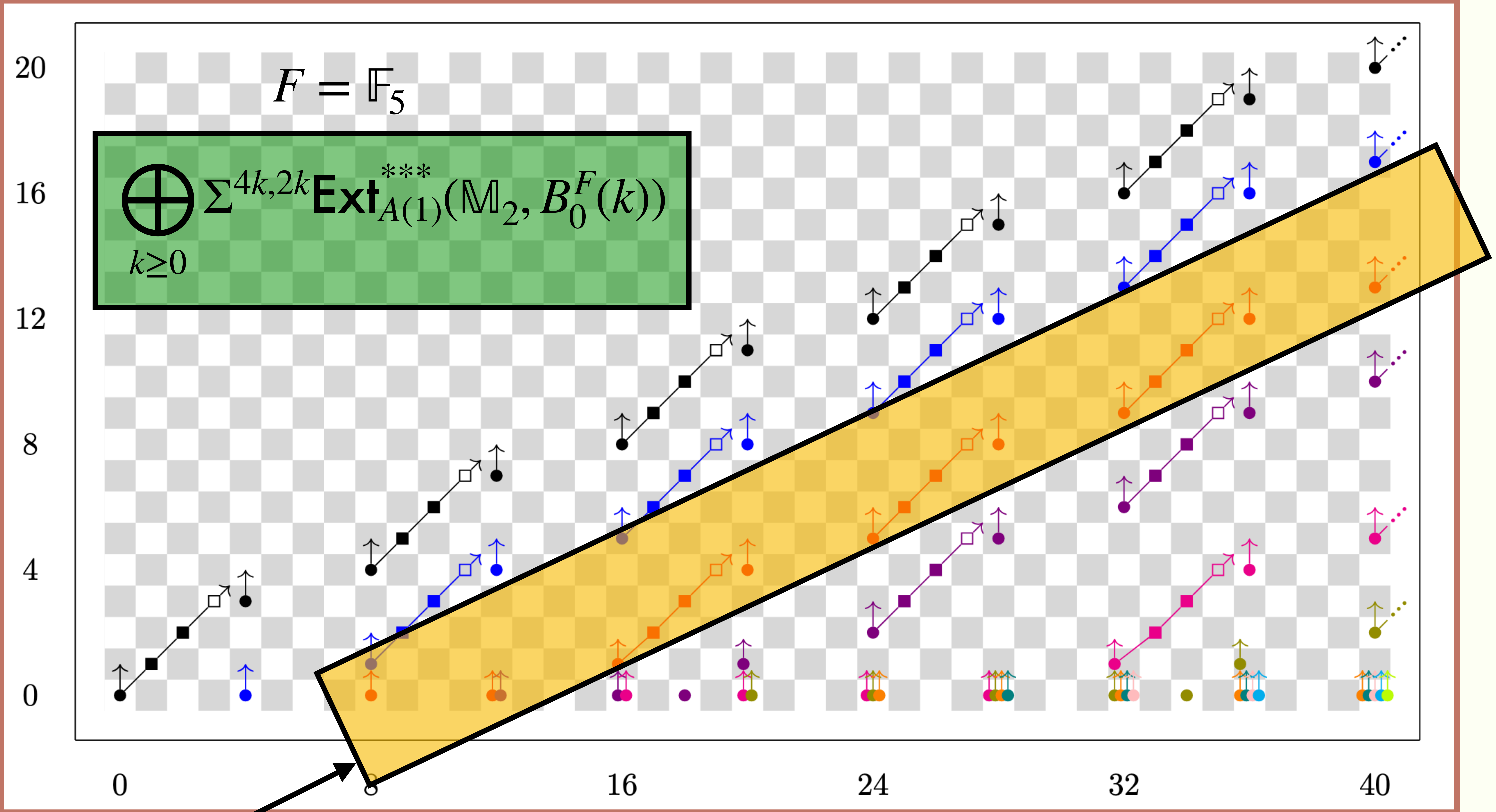


<u>Thm</u>[M.]
This is the AdamsSS for ksp!

$\mathrm{Ext}^{***}_{A(1)}(\mathbb{M}_2, B_0^F(1))$

Each color represents a different summand

Ok, but what do we get at the end?!



$\mathrm{Ext}^{***}_{A(1)}(\mathbb{M}_2, B_0^F(2))$

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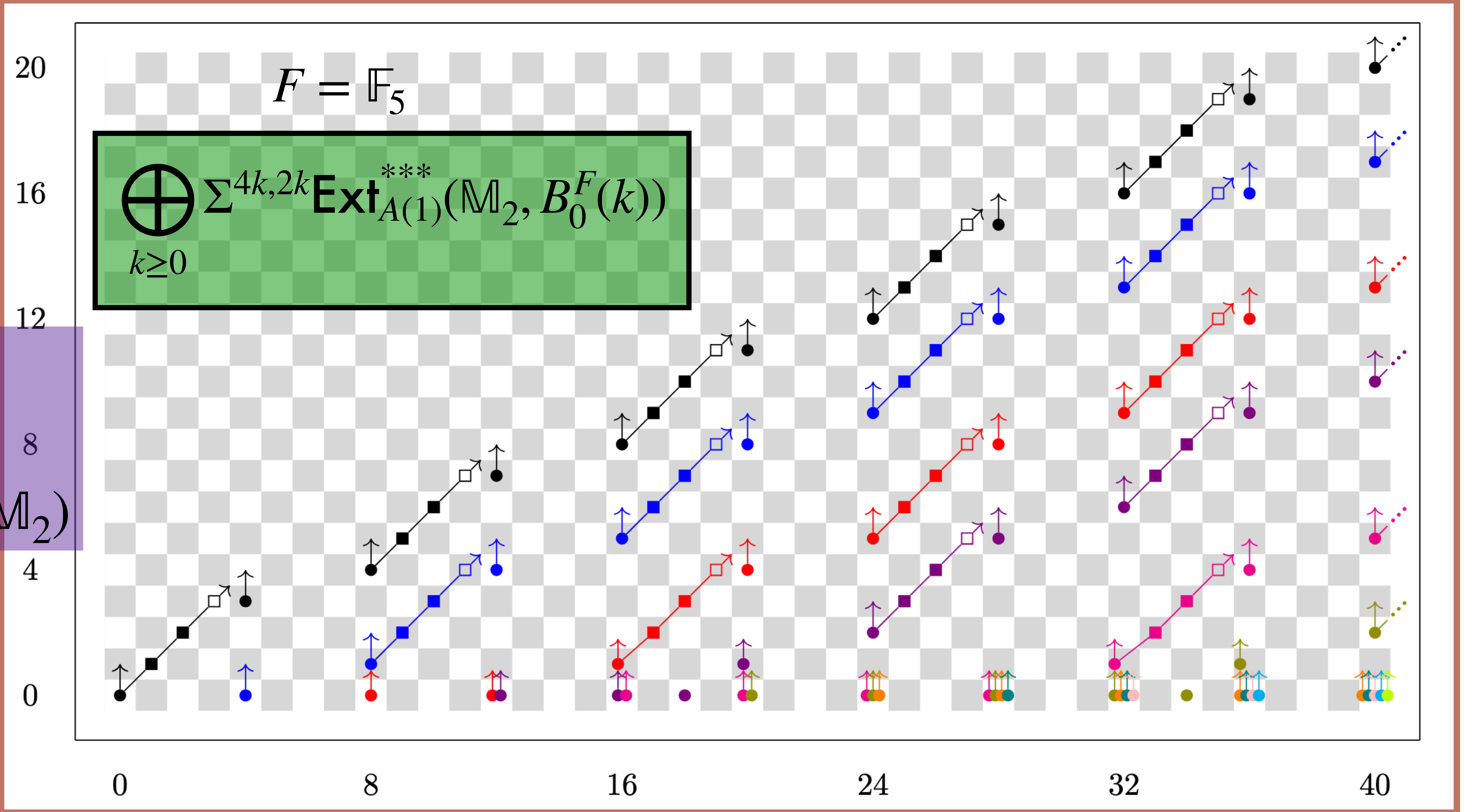
Ok, but what do we get at the end?!



The summands all look like a slightly modified $\mathrm{Ext}^{***}_{A(1)}(\mathbb{M}_2, \mathbb{M}_2)$

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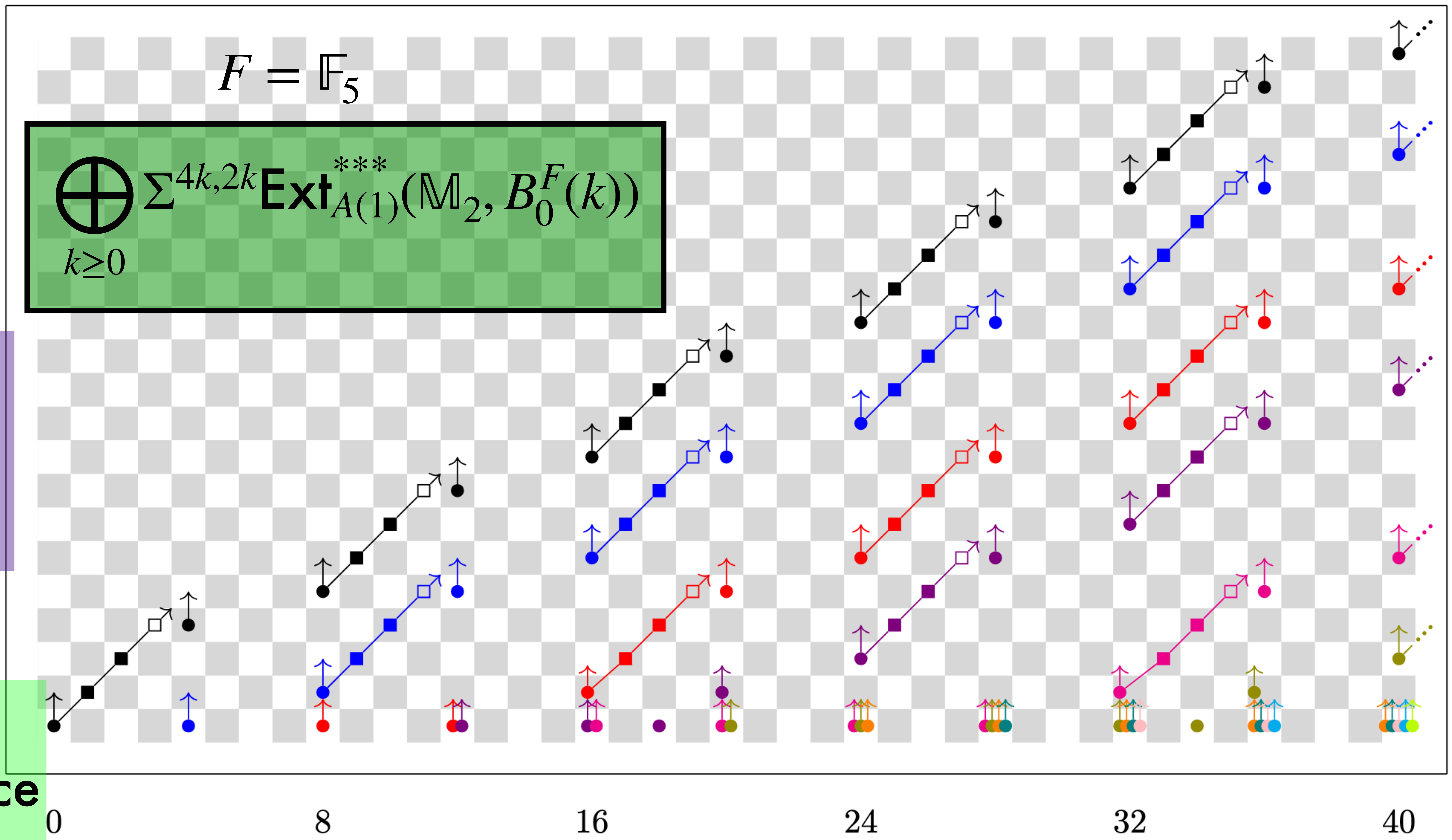

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We know how the Adams spectral sequence works here!

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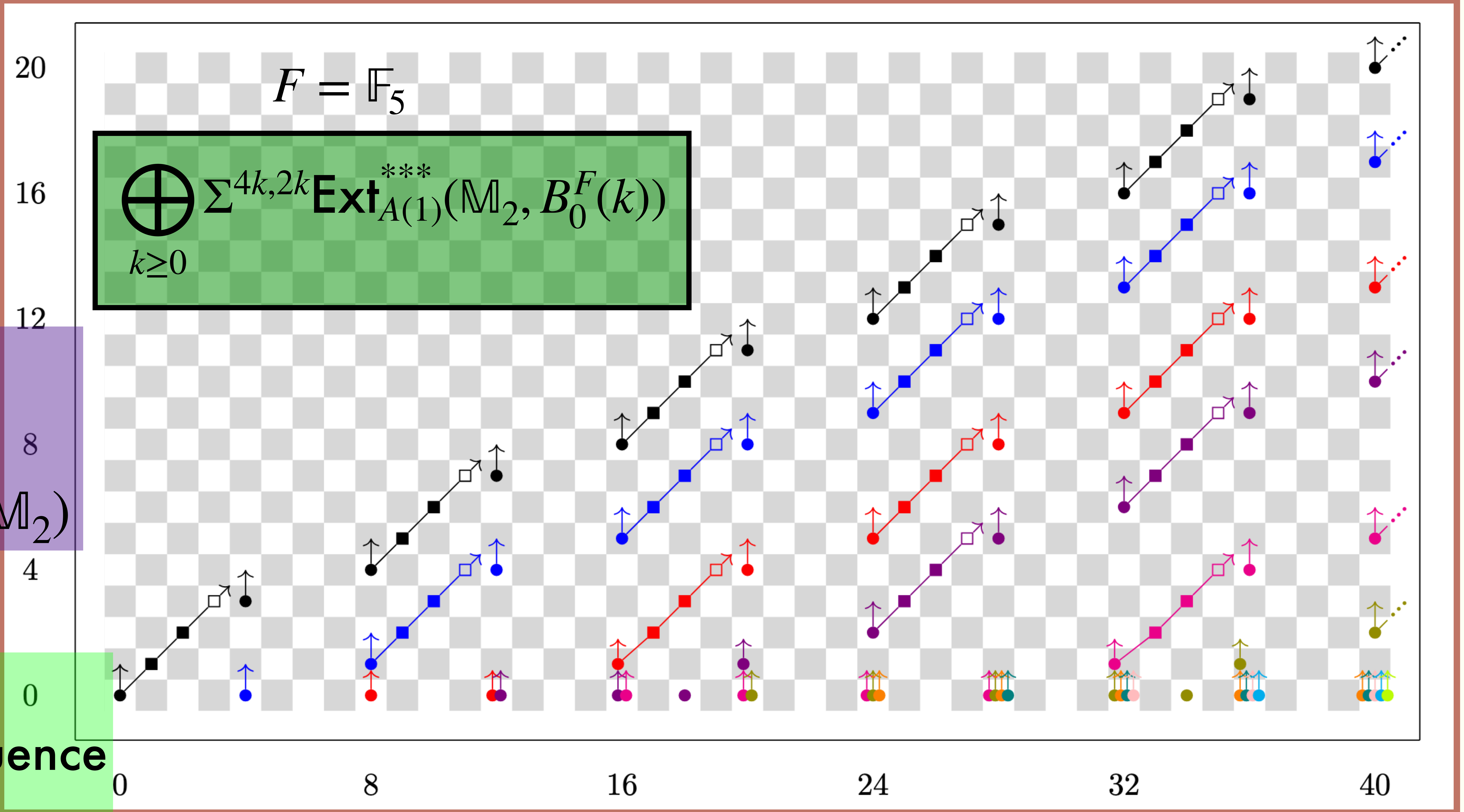

The summands all look like a slightly modified $\mathrm{Ext}^{***}_{A(1)}(\mathbb{M}_2, \mathbb{M}_2)$

We know how the Adams spectral sequence works here!

This is the key.

Each color represents a different summand

Thm[M.]

The Adams spectral sequence

$$\mathrm{Ext}_A^{***}(\mathbb{M}_2, \mathrm{H}_{**}(\mathbf{kq} \otimes \mathbf{kq})) \implies \pi^F_{**}(\mathbf{kq} \otimes \mathbf{kq})$$

is determined by the Adams spectral sequence

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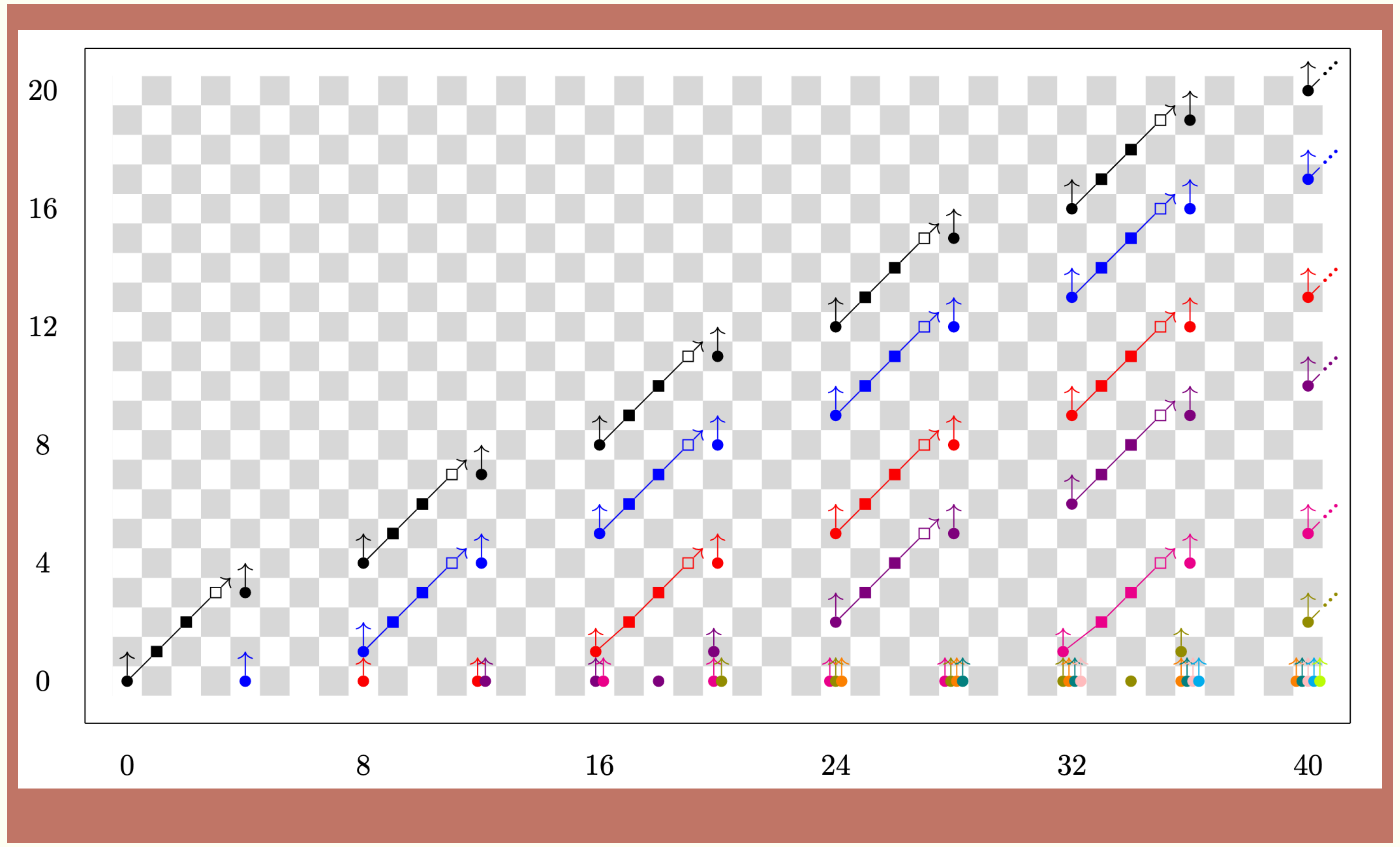

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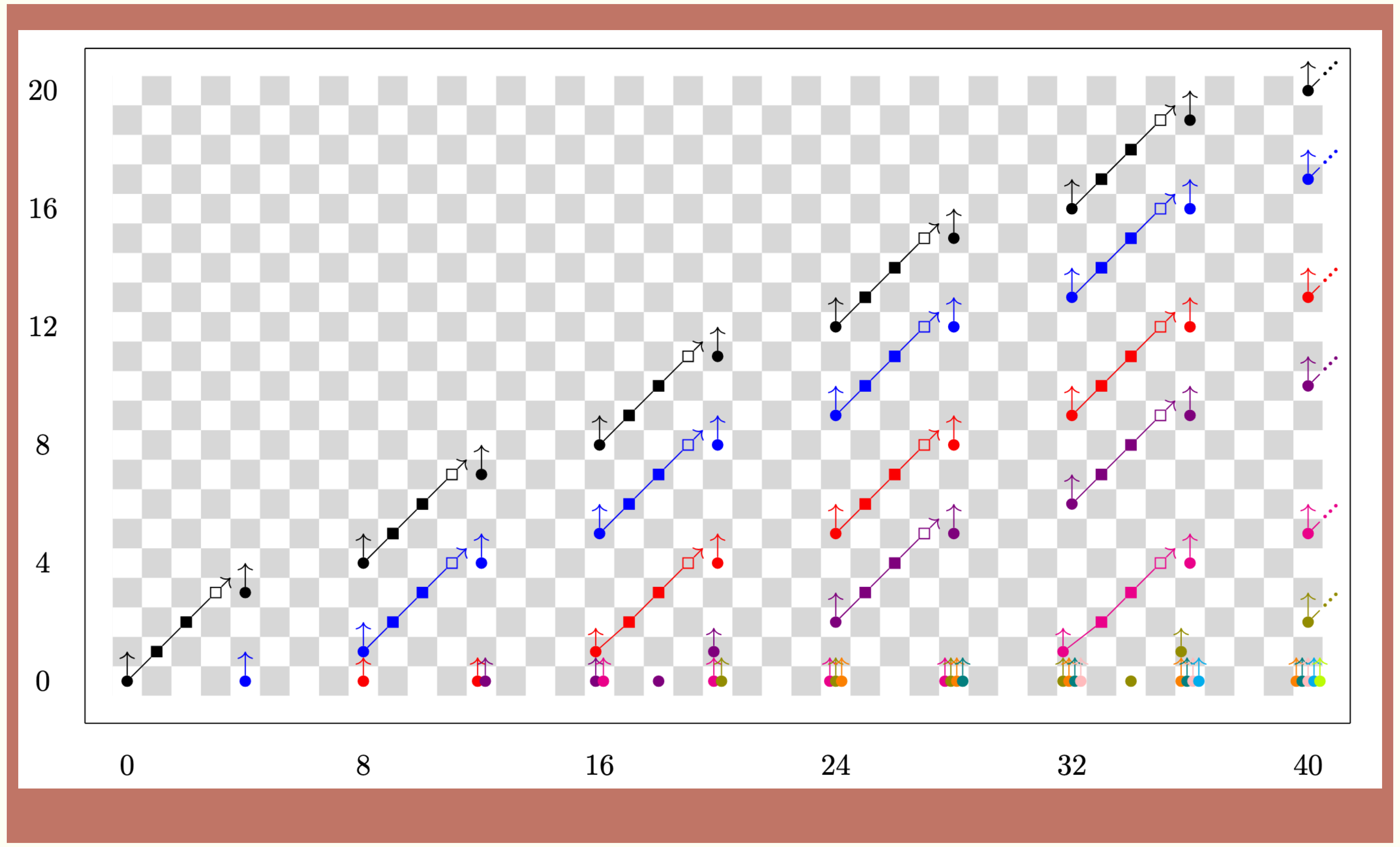
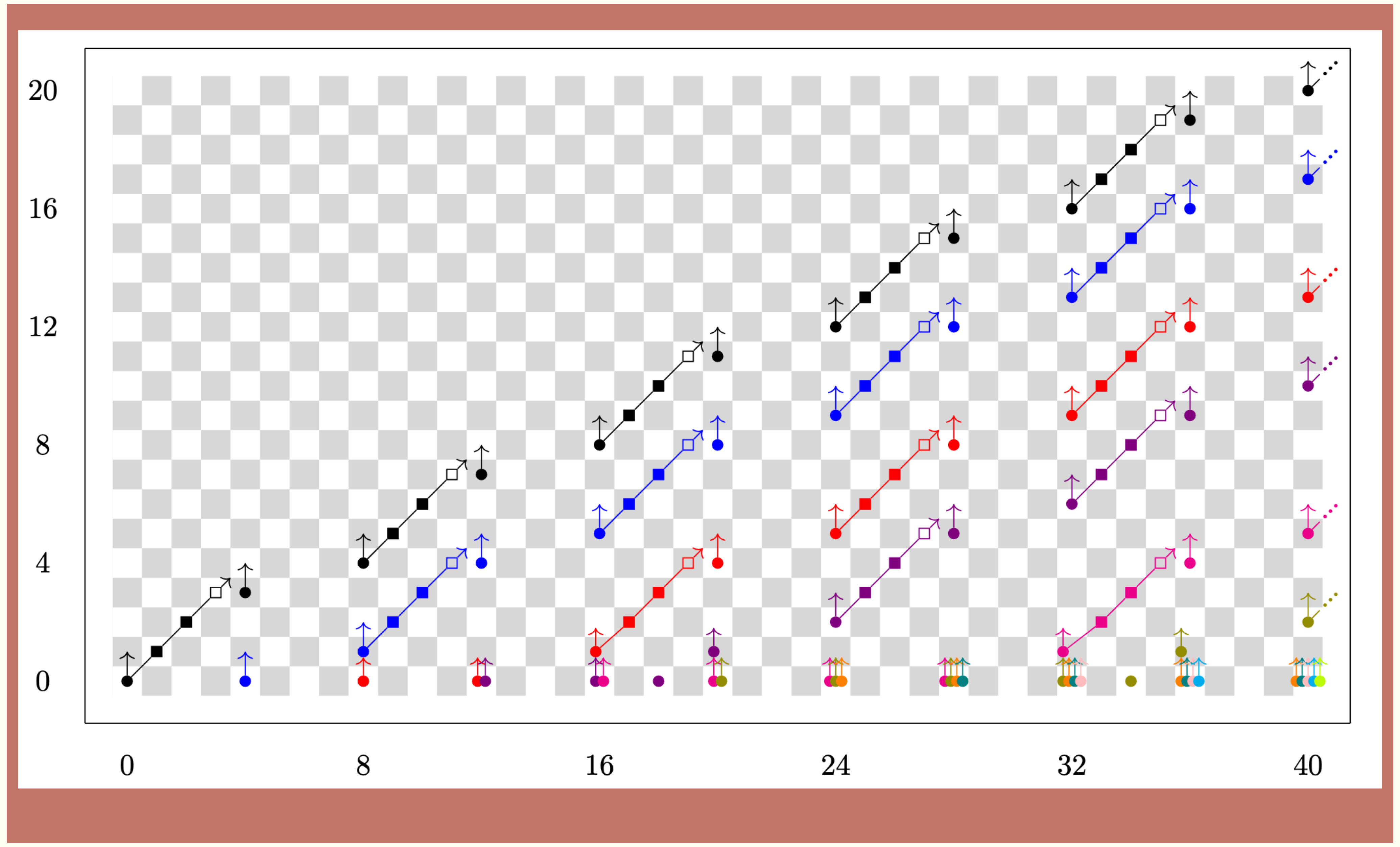


For $F = \mathbb{R}$, this implies that the spectral sequence collapses immediately



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For $F = \mathbb{F}_q$, this implies that the spectral sequence doesn't collapse at any finite page

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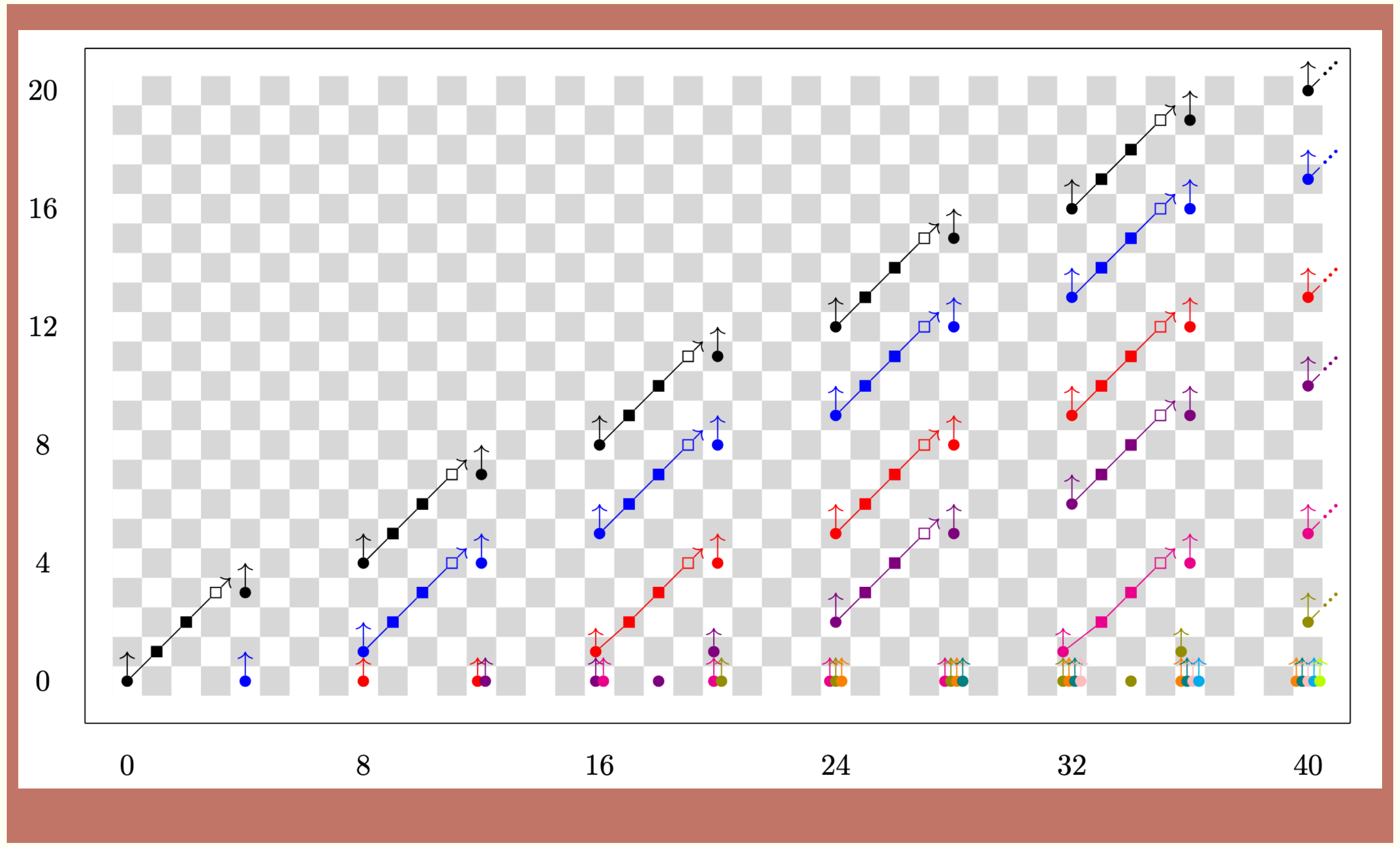

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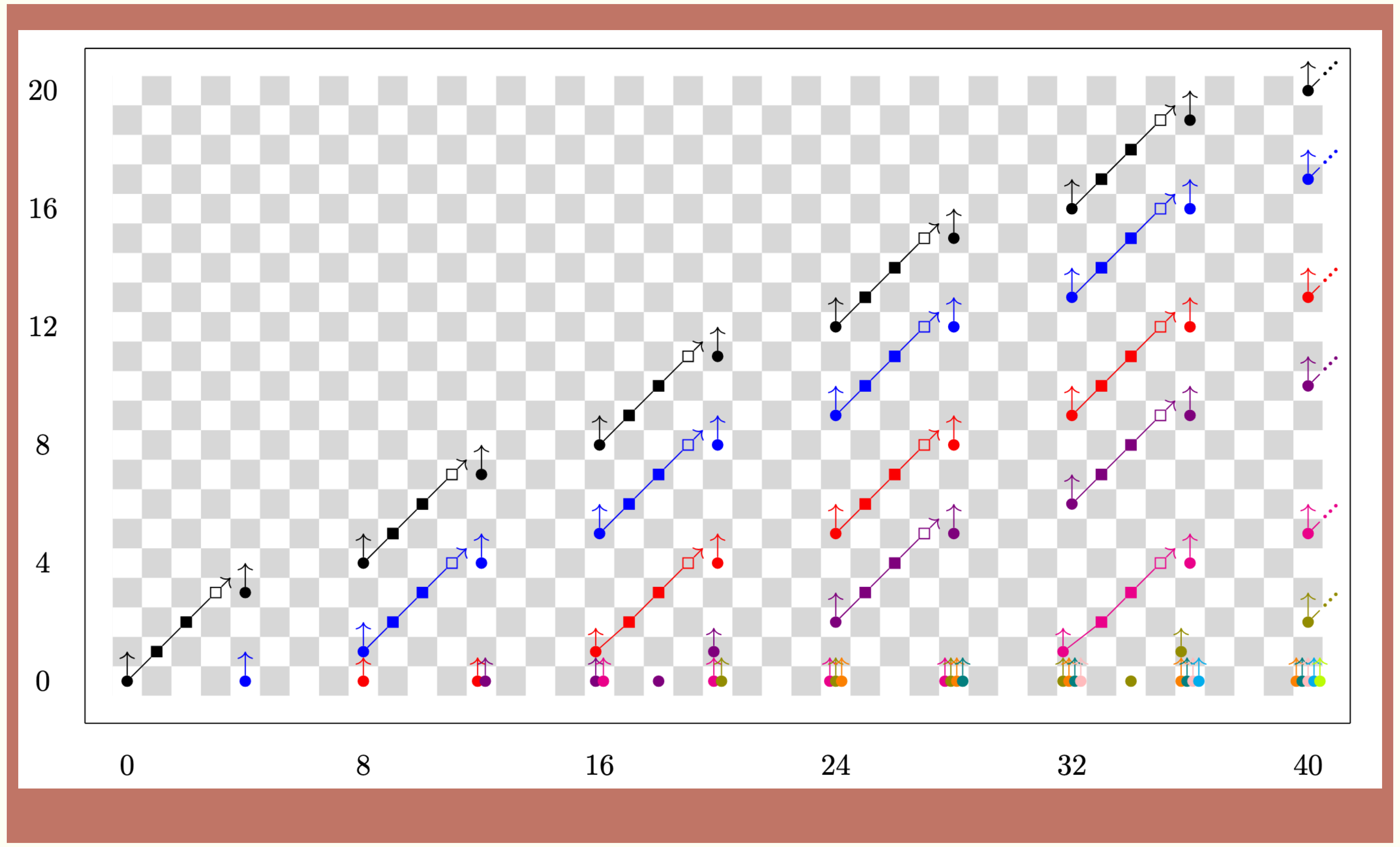

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The exact same methods give a stronger result

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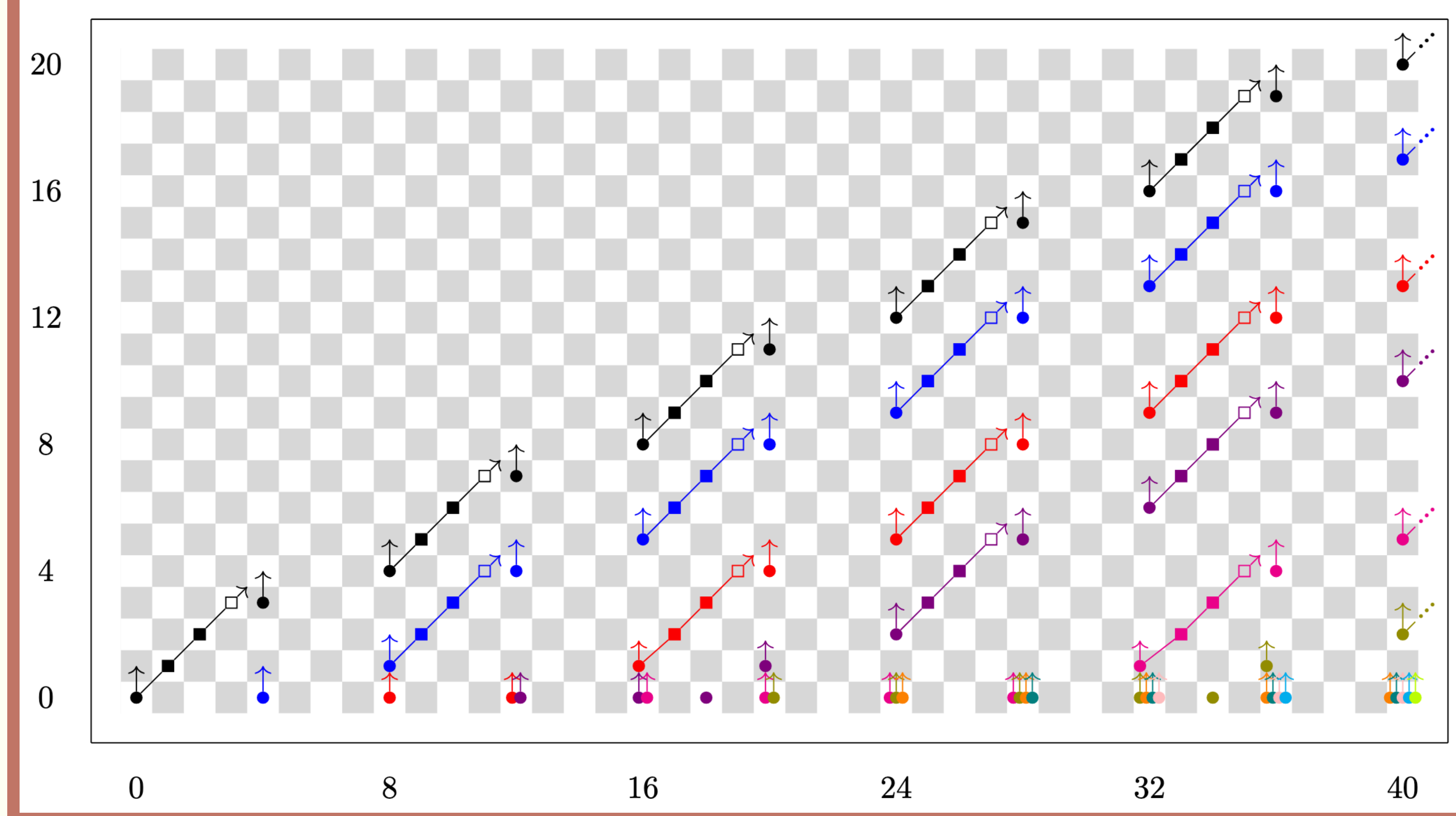

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Cor[M.]

The E_1-page of the kq-resolution

$$\pi^F_{s+f,w}(\mathbf{kq}^{\otimes f}) \implies \pi^F_{s,w}(\mathbb{S})$$

is completely understood.

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joint work with
Petersen-Tatum [2025]
(arXiv:2509.19542)

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Compute the kq-resolution!

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Hopefully soon :)

THANK YOU!