SPLITTINGS AND THE ALGEBRAIC ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE

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Abstract. These are an expanded set of notes for a pretalk on computing cooperation algebras using the algebraic Atiyah-Hirzebruch spectral sequence and algebraic splittings.

Contents

1. Introduction

Probably the best tool we have for computing stable homotopy groups is the E-based Adams spectral sequence, denoted here the E-$\mathbf{ASS}(\mathbb{S})$, where E is some connective ring spectrum. In a particular light, one may understand the E-$\mathbf{ASS}(\mathbb{S})$ as the vehicle which realizes descent along the unit map $\mathbb{S} \to \mathrm{E}$, translating information about Mod(E) into information about $\mathrm{Mod}(\mathbb{S}) \simeq \mathrm{Sp}$. This spectral sequence arises by applying $\pi_*(-)$ to the cosimplicial ring spectrum

$$\mathbb{S} \longrightarrow E \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; E \otimes \overline{E} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; E \otimes \overline{E} \otimes \overline{E} \longrightarrow \cdots$$

and has E_1-page given by

$$E_1^{s,f} = \pi_{s+f}(\mathrm{E} \otimes \overline{\mathrm{E}}^{\otimes f}) \implies \pi_s \mathbb{S}^\wedge_E,$$

where $\mathbb{S}^\wedge_E$ denotes the E-nilpotent completion [Bou79].

The point of this note is to investigate a useful method for decomposing the computation of the ring of cooperations $\pi_*(\mathrm{E} \otimes \mathrm{E})$ into more manageable and computable pieces. We will indicate how one can extend these ideas to compute the E_1-page of the E-$\mathbf{ASS}(\mathbb{S})$.

2. Flat rings

Suppose that E is a flat ring spectrum, meaning that $\pi_*(\mathrm{E} \otimes \mathrm{E})$ is flat as a module over $\pi_*\mathrm{E}$. In this case, we may identify the E_1-page of the Adams spectral sequence with the cobar complex $C_{\pi_*\mathrm{E}}(\pi_*(\mathrm{E}\otimes\mathrm{E}))$. In particular, this leads to a description of the E_2-page as Hopf algebroid cohomology:

$$E_2 = \mathrm{Ext}_{\pi_*(\mathrm{E}\otimes\mathrm{E})}(\pi_*\mathrm{E}, \pi_*\mathrm{E}).$$

The ring of cooperations appears here as the Hopf algebroid whose cohomology we can use to eek some information out of the sphere. In some cases, we do not have to do much to compute this.

Example 2.1. Let $\mathrm{E} = \mathrm{MU}$ or BP. Using the isomorphisms

$$\mathrm{H}\mathbb{Z}_*(\mathrm{MU}) \cong \mathbb{Z}[a_1, a_2, \dots] \quad \text{and} \quad \mathrm{H}\mathbb{F}_{2*}(\mathrm{BP}) \cong \mathbb{F}_2[\xi_1^2, \xi_2^2, \dots],$$

one can show that the relevant Atiyah-Hirzebruch spectral sequences computing $\pi_*(\mathrm{MU} \otimes \mathrm{MU})$ and $\pi_*(\mathrm{BP} \otimes \mathrm{BP})$ must collapse, giving isomorphisms

$$\pi_*(\mathrm{MU} \otimes \mathrm{MU}) \cong (\pi_*\mathrm{MU})[b_1, b_2, \dots], \quad \text{and} \quad \pi_*(\mathrm{BP} \otimes \mathrm{BP}) \cong (\pi_*\mathrm{BP})[t_1, t_2, \dots].$$

Having an understanding on these rings of cooperations is a large part of why we can even begin to understand the algebra behind the Adams-Novikov spectral sequence.

However, in even some small cases it is not so easy.

Example 2.2. Let $\mathrm{E} = \mathrm{H}\mathbb{F}_p$. Then $\pi_*(\mathrm{H}\mathbb{F}_p \otimes \mathrm{H}\mathbb{F}_p) = \mathcal{A}^\vee$ is the dual Steenrod algebra. This is not the easiest thing to determine. The usual reference is [Mil58], although there may be a stacky/DAG/SAG/weird way to determine the structure of the dual Steenrod algebra which one might find in any of the set of chromatic homotopy theory notes of Hopkins [Hop99], Lurie [Lur10], or Pstragowski [Pst21]. Regardless, the structure is complicated:

$$\mathcal{A}_p^\vee \cong \mathbb{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \dots)$$

for p odd, and

$$\mathcal{A}_2^\vee \cong \mathbb{F}_2[\zeta_1, \zeta_2, \dots].$$

For comodule action reasons to appear later, we are using the basis given by the conjugate elements $\zeta_i := \overline{\xi}_i$. While these algebras are complicated, they are also well studied. I mean, the resulting Adams spectral sequence is called *the* Adams spectral sequence for a reason.

Often, the Adams spectral sequence is presented based on a flat ring spectrum, such as the ones given above. Flatness should really be thought of as a rare phenomena, though, and not one which we should rely on. So, the question is brought up: How does one compute the E_1-page of the Adams spectral sequence, or even the ring of cooperations, for a general ring spectrum?

3. The Adams spectral sequence for an Adams spectral sequence

For the rest of this note, we will be working at the prime 2, hence all of our results will be implicitly 2-complete, the dual Steenrod algebra will be 2-primary, and all homology will be mod-2. One way that we can compute the ring of cooperations is by an Adams spectral sequence:

$$E_2 = \mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, \mathrm{H}_*(\mathrm{E} \otimes \mathrm{E})) \implies \pi_*(\mathrm{E} \otimes \mathrm{E}).$$

We're tasked now with understanding the homology $\mathrm{H}_*(\mathrm{E} \otimes \mathrm{E})$ as a comodule over $\mathcal{A}^\vee$. However, we can use the Künneth isomorphism to turn this problem into understanding the comodule structure of $\mathrm{H}_*(\mathrm{E}) \otimes \mathrm{H}_*(\mathrm{E})$.

It is at this point that we specialize our spectra to ones with a nice property. To preface what we are going to do, I want to put a general philosophy into your mind:

To solve a big problem, break it up into smaller pieces, then break those smaller pieces into even smaller pieces, and then break those even smaller pieces into ...

We've already been practicing this. To compute the E_1-page of the $\mathrm{E}-\mathbf{ASS}(\mathbb{S})$, we are bootstrapping our way up from the ring of cooperations. To compute the ring of cooperations, we are bootstrapping our way up from the homology $\mathrm{H}_*(\mathrm{E} \otimes \mathrm{E})$. To understand this homology, we can use the Künneth isomorphism and study $\mathrm{H}_*(\mathrm{E}) \otimes \mathrm{H}_*(\mathrm{E})$.

The point now is to keep breaking this problem into smaller, more manageable problems that we can sink our baby teeth into. So, instead of understanding the structure of $\mathrm{H}_*(\mathrm{E}) \otimes \mathrm{H}_*(\mathrm{E})$ as a comodule over $\mathcal{A}^\vee$, we should try to find a way to break the problem down further.

Remark 3.1. Although we are specializing to the category $\mathrm{CoMod}(\mathcal{A}^\vee)$, the general framework of what we have described in the previous section and what we will describe in the latter sections is conceptually useful in other categories. For instance, instead of using the mod-2 Adams spectral sequence to study $\pi_*(\mathrm{E}\otimes\mathrm{E})$, we also could've used the Adams Novikov:

$$\mathrm{Ext}_{\pi_*(\mathrm{BP}\otimes\mathrm{BP})}(\pi_*\mathrm{BP}, \mathrm{BP}_*(\mathrm{E}\otimes\mathrm{E})) \implies \pi_*((\mathrm{E}\otimes\mathrm{E})_{(p)}).$$

However, BP does not in general have a Künneth isomorphism, and we can't continue our philosophy of breaking the problem into more bite-sized pieces; we have to teak a big gulp and work hard to understand $\mathrm{BP}_*(\mathrm{E}\otimes\mathrm{E})$. The same happens when we work with the MU-Adams Novikov.

One attempt to fix this issue is to try and work with a flat ring spectrum which admits a Künneth isomorphism. One example of such a spectrum is Morava K-theory $\mathrm{K}(n)$. The spectral sequence we'd be looking at, however, takes the form

$$\mathrm{Ext}_{\pi_*(\mathrm{K}(n)\otimes\mathrm{K}(n))}\left(\mathbb{F}_p[v_n^{\pm1}], \mathrm{K}(n)_*(\mathrm{E})\otimes\mathrm{K}(n)_*(\mathrm{E})\right) \implies \pi_*\mathrm{L}_{\mathrm{K}(n)}(\mathrm{E}\otimes\mathrm{E}).$$

The spectral sequence converges to the $\mathrm{K}(n)$-localization of the ring of cooperations. Unless E is $\mathrm{K}(n)$-local, this is quite a bit further away from $\pi_*(\mathrm{E}\otimes\mathrm{E})$ than the p-completion, given by the mod-p Adams spectral sequence, or the p-localization, given by the Adams-Novikov. Additionally, since $\mathrm{K}(n)$ isn't connective, the spectral sequence doesn't have the same convergence properties as our earlier, more well-behaved ones. Even if it does converge, we are working now with a half-plane spectral sequence instead of a first quadrant spectral sequence. One could try to fix this by working with the connective cover $\mathrm{k}(n)$, but now we lose our Künneth isomorphism since $\mathrm{k}(n)$ isn't a field spectrum (although the Künneth spectral sequence is not bad), and we don't (maybe it's just for me) understand the $\mathrm{k}(n)$-nilpotent completion as well.

4. The change of rings isomorphism

There are a few families of algebras which are related to the dual Steenrod algebra that we will use.

- The subalgebra $(\mathcal{A}\,/\!/\,\mathcal{A}(n))^\vee = \mathbb{F}_2[\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, \dots, \zeta_{n+1}^2, \zeta_{n+2}, \zeta_{n+3}, \dots]$;
- The quotient algebra $\mathcal{A}(n)^\vee = \mathbb{F}_2[\zeta_1, \dots, \zeta_{n+1}]/(\zeta_1^{2^{n+1}}, \dots, \zeta_{n+1}^2)$;
- The subalgebra $(\mathcal{A}\,/\!/\,\mathcal{E}(n))^\vee = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \dots, \zeta_{n+1}^2, \zeta_{n+2}, \zeta_{n+3}, \dots]$;
- The quotient algebra $\mathcal{E}(n)^\vee = \mathbb{F}_2[\zeta_1, \dots, \zeta_{n+1}]/(\zeta_1^2, \dots, \zeta_{n+1}^2$.

The interest in these families is that they give rise to an algebraic change of rings isomorphism in Ext. There are isomorphisms (which are sometimes called "derived cotensor-Hom adjuntions"):

$$\mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, (\mathcal{A}\,/\!/\,\mathcal{A}(n))^\vee) \cong \mathrm{Ext}_{\mathcal{A}(n)^\vee}(\mathbb{F}_2, \mathbb{F}_2);$$

$$\mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, (\mathcal{A}\,/\!/\,\mathcal{E}(n))^\vee) \cong \mathrm{Ext}_{\mathcal{E}(n)^\vee}(\mathbb{F}_2, \mathbb{F}_2).$$

For proofs, see [Rav86].

In fact, these isomorphisms in Ext can be realized in the real world (by real world I mean as the homology of a spectrum: none of this is real, it's math!).

Example 4.1. Notice that $(\mathcal{A}\,/\!/\,\mathcal{A}(0))^\vee \cong (\mathcal{A}\,/\!/\,\mathcal{E}(0))^\vee$. A spectrum which realizes this algebra in homology is $\mathrm{H}\mathbb{Z}$:

$$\mathrm{H}_*(\mathrm{H}\mathbb{Z}) \cong (\mathcal{A}\,/\!/\,\mathcal{A}(0))^\vee.$$

Example 4.2. Let $\mathrm{BP}\langle n\rangle = \mathrm{BP}/(v_{n+1}, v_{n+2}, \dots)$. Then there is an isomorphism of $\mathcal{A}^\vee$-comodules:

$$\mathrm{H}_*(\mathrm{BP}\langle n\rangle) \cong (\mathcal{A}\,/\!/\,\mathcal{E}(n))^\vee.$$

For $n = 0$, this recovers the previous example. For $n = 1$, a model for $\mathrm{BP}\langle 1\rangle$ is $\mathrm{bu} = \tau_{\geq 0}\mathrm{KU}$, the connective cover of complex K-theory (using Mahowald's notation instead of ku, sorry). For $n = 2$, a model for $\mathrm{BP}\langle 2\rangle$ is $\mathrm{tmf}_1(3)$.

Example 4.3. The spectrum $\mathrm{bo} = \tau_{\geq 0}\mathrm{KO}$, the connective cover of real K-theory, has homology

$$\mathrm{H}_*(\mathrm{bo}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee.$$

The spectrum tmf has homology

$$\mathrm{H}_*(\mathrm{tmf}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}(2))^\vee.$$

There is a slick argument to prove that there is no spectrum realizing $(\mathcal{A} \,/\!/\, \mathcal{A}(n))^\vee$ in homology for $n \geq 3$ by showing that if such a spectrum exists, then it would break the Hopf invariant 1 solution of Adams.

In short, what this buys us is that if we want to compute the ring of cooperations $\pi_*(\mathrm{E} \otimes \mathrm{E})$ for $\mathrm{E} \in \{\mathrm{BP}\langle n\rangle, \mathrm{bo}, \mathrm{tmf}\}$, then we can use a change of rings isomorphism to go that "one step further" in our philosophy. In the case of $\mathrm{BP}\langle n\rangle$, the E_2-page of the Adams spectral sequence takes the form

$$\mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, \mathrm{H}_*(\mathrm{BP}\langle n\rangle) \otimes \mathrm{H}_*(\mathrm{BP}\langle n\rangle) \cong \mathrm{Ext}_{\mathcal{E}(n)^\vee}(\mathbb{F}_2, \mathrm{H}_*(\mathrm{BP}\langle n\rangle)).$$

In the case of bo, the E_2-page of the Adams spectral sequence takes the form

$$\mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, \mathrm{H}_*(\mathrm{bo}) \otimes \mathrm{H}_*(\mathrm{bo})) \cong \mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, \mathrm{H}_*(\mathrm{bo})).$$

A similar story holds for tmf, but I'm not going to talk about that case much (for more details, see [BOSS19, Cul19, Tat23]). In each case, we have reduced to studying the homology of a spectrum as a comodule over a now finite-dimensional algebra. This rules! But again, both $\mathrm{H}_*(\mathrm{bo})$ and $\mathrm{H}_*(\mathrm{BP}\langle n\rangle)$ are very large comodules. It is hard to chew these pieces of algebra without taking out our knife and cutting things up a little more. I point to our philosophy once more: we should break up the homology of these spectra into smaller pieces.

5. Brown-Gitler comodules and Algebraic splittings

There is a simple to define weight filtration on the dual Steenrod algebra, attributed to Mahowald and often called the Mahowald filtration. Let $wt(\zeta_i) = 2^i$, and set $wt(xy) = wt(x) + wt(y)$. This filtration naturally extends to the subalgebras $(\mathcal{A} \,/\!/\, \mathcal{A}(n))^\vee$. Moreover, this is a filtration by comodules, meaning that the coaction of $\mathcal{A}^\vee$ (or its various quotients $\mathcal{A}(n)^\vee$) on $\mathcal{A}^\vee$ (or its various subalgebras $(\mathcal{A} \,/\!/\, \mathcal{A}(n))^\vee$ and $(\mathcal{A} \,/\!/\, \mathcal{E}(n))^\vee$) preserve the Mahowald weight. This allows one to define what are called the *Brown-Gitler comodules*:

$$B_{\mathcal{A}(n)^\vee}(k) = \langle x \in (\mathcal{A} \,/\!/\, \mathcal{A}(n))^\vee \ : wt(x) \leq 2^{n+1}k\rangle,$$

$$B_{\mathcal{E}(n)^\vee}(k) = \langle x \in (\mathcal{A} \,/\!/\, \mathcal{E}(n))^\vee : wt(x) \leq 2^{n+1}k\rangle.$$

For $n = 0$, the two defintions agree since $(\mathcal{A} \,/\!/\, \mathcal{E}(0))^\vee \cong (\mathcal{A} \,/\!/\, \mathcal{A}(0))^\vee$. We denote the comodules in this case by $B_0(k)$, and they are called the integral Brown-Gitler comodules. We'll work with this family the most; here are two examples.

Example 5.1. The Brown-Gitler comodule $B_0(0)$ is given by

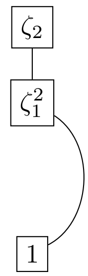
$$B_0(0) = \langle x \in (\mathcal{A} \,/\!/\, \mathcal{A}(0))^\vee : wt(x) \leq 0\rangle \cong \mathbb{F}_2,$$

with trivial $\mathcal{A}(1)^\vee$-comodule structure.

Example 5.2. The Brown-Gitler comodule $B_0(1)$ is given by

$$B_0(1) = \langle x \in (\mathcal{A} \,/\!/\, \mathcal{A}(0))^\vee : wt(x) \leq 2\rangle = \mathbb{F}_2\{1, \zeta_1^2, \zeta_2\},$$

which we can represent as the following algebraic cell complex.



For $n = 1$, the comodules $B_{\mathcal{A}(1)^\vee}(k)$ are called the bo Brown-Gitler comodules. These names come from the equivalences:

$$\mathrm{colim}_k(B_0(k)) = (\mathcal{A} \,/\!/\, \mathcal{A}(0))^\vee = \mathrm{H}_*(\mathrm{H}\mathbb{Z}), \quad \mathrm{colim}_k(B_{\mathcal{A}(1)^\vee}(k)) = (\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee = \mathrm{H}_*(\mathrm{bo}).$$

The analogy holds true for $n = 2$, but I'm not really going to discuss this case so I won't write out the details.

The point of these comodules is that not only do they filter homology as listed above, but they also assemble to additive splittings. In particular, there are is an isomorphism of $\mathcal{A}(1)^\vee$-comodules

$$\mathrm{H}_*(\mathrm{bo}) \cong \bigoplus_{k\geq 0} \Sigma^{4k} B_0(k),$$

and isomorphisms of $\mathcal{E}(n)^\vee$-comodules

$$\mathrm{H}_*(\mathrm{BP}\langle n\rangle) \cong \bigoplus_{k\geq 0} \Sigma^{2k} B_{\mathcal{E}(n-1)^\vee}(k).$$

This allows us to decompose our Adams spectral sequences one step further. We have isomorphisms

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, \mathrm{H}_*(\mathrm{bo})) \cong \bigoplus_{k\geq 0} \Sigma^{4k} \mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, B_0(k)),$$

and

$$\mathrm{Ext}_{\mathcal{E}(n)^\vee}(\mathbb{F}_2, \mathrm{H}_*(\mathrm{BP}\langle n\rangle)) \cong \bigoplus_{k\geq 0} \Sigma^{2k} \mathrm{Ext}_{\mathcal{E}(n)^\vee}(\mathbb{F}_2, B_{\mathcal{E}(n-1)^\vee}(k)).$$

Great! We've cut our problem into bite-sized pieces. But I'm a little bit of a freak: I want to be able to swallow the pieces whole. For that, we need to go one step further.

Remark 5.3. Some of the Brown-Gitler comodules are realized as the homology of the so-called *Brown-Gitler spectra*. There are integral Brown-Gitler spectra $\mathrm{H}\mathbb{Z}_k$ which filter $\mathrm{H}\mathbb{Z}$ and have homology given by $\mathrm{H}_*(\mathrm{H}\mathbb{Z}_k) = B_0(k)$. There are bo Brown-Gitler spectra bo_k which filter bo and have homology given by $\mathrm{H}_*(\mathrm{bo}_k) = B_{\mathcal{A}(1)^\vee}(k)$. There are bu Brown-Gitler spectra bu_k which filter bu and have homology given by $\mathrm{H}_*(\mathrm{bu}_k) = B_{\mathcal{E}(1)^\vee}(1)$. Moreover, these spectra often give rise to topological splittings. For instance,

$$\mathrm{bu} \otimes \mathrm{bu} \simeq \bigoplus_{k\geq 0} \Sigma^{2k} \mathrm{bu} \otimes \mathrm{H}\mathbb{Z}_k,$$

$$\mathrm{BP}\langle 2\rangle \otimes \mathrm{BP}\langle 2\rangle \simeq \bigoplus_{k\geq 0} \Sigma^{2k} \mathrm{BP}\langle 2\rangle \otimes \mathrm{bu}_k,$$

$$\mathrm{bo} \otimes \mathrm{bo} \simeq \bigoplus_{k\geq 0} \Sigma^{4k} \mathrm{bo} \otimes \mathrm{H}\mathbb{Z}_k;$$

see [Ada74], [Tat23], and [Mah81] respectively. However, there are no constructions of these spectra for arbitrary n, and even still, we do not always have the splitting one might expect. For instance, it was shown by Davis, Mahowald, and Rezk that

$$\mathrm{tmf} \otimes \mathrm{tmf} \not\simeq \bigoplus_{k\geq 0} \Sigma^{8k} \mathrm{tmf} \otimes \mathrm{bo}_k.$$

In the presence of topological splittings, one can try to compute the ring of cooperations in a way different than how we will present. To be as general as possible, we will proceed in an algebraic manner.

6. The algebraic Atiyah-Hirzebruch spectral sequence

The question towards the ring of cooperations has become: how do we compute the groups $\mathrm{Ext}_{\mathrm{B}}(\mathbb{F}_2, B_{\mathrm{B}}(k))$ for $\mathrm{B} \in \{\mathcal{A}(1)^\vee, \mathcal{E}(n)^\vee\}$? From the definitions, we see that $B_{\mathrm{B}}(0) = \mathbb{F}_2$ for all values of n and all B. The corresponding Ext groups come up in nature: $\mathrm{Ext}_{\mathcal{E}(n)^\vee}(\mathbb{F}_2, \mathbb{F}_2)$ is the E_2-page of the Adams spectral sequence computing $\pi_*\mathrm{BP}\langle n\rangle$, and $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2)$ is the E_2-page of the Adams spectral sequence computing $\pi_*\mathrm{bo}$. We can compute these fairly easily using either the Cobar complex or the May spectral sequence (see [Rav86, Chapter 3]). We have:

$$\mathrm{Ext}_{\mathcal{E}(n)^\vee}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[v_0, v_1, \ldots, v_n]$$

and

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, a, b]/(h_0h_1, h_1^3, h_1a, a^2 + h_0^2 b).$$

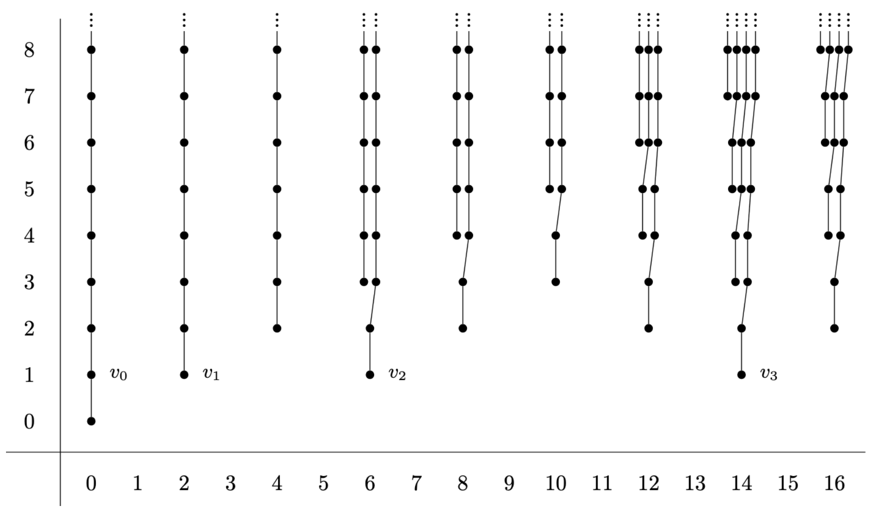


Figure 1. $\mathrm{Ext}_{\mathcal{E}(n)^\vee}(\mathbb{F}_2, \mathbb{F}_2)$

To get to $k = 1$, we can leverage our algebra a little. Take the Brown-Gitler comodule $B_{\mathrm{B}}(1)$ and filter by degree, using the grading of $\mathcal{A}^\vee$. This gives a finite filtration:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{F}_0 B_{\mathrm{B}}(1) \arrow[r] \arrow[d] & \mathrm{F}_1 B_{\mathrm{B}}(1) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{F}_k B_{\mathrm{B}}(1) = B_{\mathrm{B}}(1) \arrow[d] \\ & \mathrm{gr}_0^{\mathrm{cell}} B_{\mathrm{B}}(1) & \mathrm{gr}_1^{\mathrm{cell}} B_{\mathrm{B}}(1) & & \mathrm{gr}_k^{\mathrm{cell}} B_{\mathrm{B}}(1) \end{tikzcd}$$

The associated graded pieces $\mathrm{gr}_m^{\mathrm{cell}} B_{\mathrm{B}}(1)$ are the monomials of $B_{\mathrm{B}}(1)$ of degree exactly m. Applying the functor $\mathrm{Ext}_{\mathrm{B}}(\mathbb{F}_2, -)$ gives the *algebraic Atiyah-Hirzebruch spectral sequence*. It takes the form

$$E_1 = \bigoplus_{0 \le i \le k} \mathrm{Ext}_{\mathrm{B}}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{gr}_i^{\mathrm{cell}} B_{\mathrm{B}}(1) \implies \mathrm{Ext}_{\mathrm{B}}(\mathbb{F}_2, B_{\mathrm{B}}(1)).$$

The differentials are given by the attaching maps/comodule structure of the Brown-Gitler comodule which is being filtered. Sometimes this means direct multiplication by an element in Ext; other times, one must compute a Massey product. At the E_∞-page, we must resolve hidden extensions using... magic. This doesn't seem like it is the most useful technique on the surface. After all, we're now like

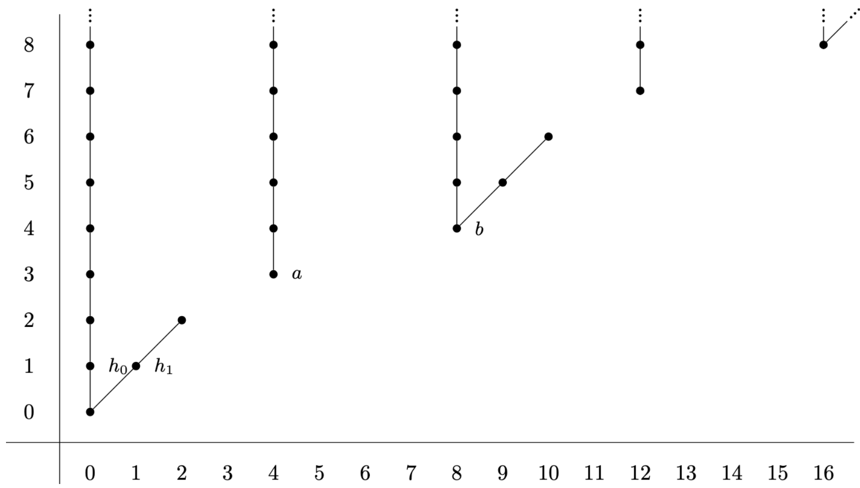


FIGURE 2. $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2)$

3 spectral sequences deep. But it turns out that this algebraic Atiyah-Hirzebruch spectral sequence is quite nice!

Let's describe the E_1-page. This looks like a bunch of copies of $\mathrm{Ext}_{\mathrm{B}}(\mathbb{F}_2, \mathbb{F}_2)$ (one of the charts given in the figures above), shifted by the degree of the cell it is attached to. Differentials always go up and to the left by 1, and down in filtration by whatever page of the spectral sequence we're on. It's a very fun game to play, and efficient once you can get your head screwed on right (this is what I do in [Mor25]). We'll go into a few examples in detail in the next section.

Remark 6.1. One could try and filter the other Brown-Gitler comodules in the same way and run the algebraic Atiyah-Hirzebruch spectral sequence to compute $\mathrm{Ext}_{\mathrm{B}}(\mathbb{F}_2, B_n(k))$ for all $k \geq 0$, and this might work. I honestly have not tried, nor do I think others have. There is another way to get the rest of our Ext groups: there are short exact sequences of Brown-Gitler comodules:

$$0 \to \Sigma^{4k} B_0(k) \to B_0(2k) \to B_{\mathcal{A}(1)^\vee}(k-1) \otimes (\mathcal{A}(1) \,/\!/\, \mathcal{A}(0))^\vee \to 0,$$

$$0 \to \Sigma^{4k} B_0(k) \otimes B_0(1) \to B_0(2k+1) \to B_{\mathcal{A}(1)^\vee}(k-1) \otimes (\mathcal{A}(1) \,/\!/\, \mathcal{A}(0))^\vee \to 0.$$

These allow us to induct along k to get our answer, fully describing the E_2-page of the Adams spectral sequence we are interested in, by studying the long exact sequence arising from applying $\mathrm{Ext}_{\mathrm{B}}(\mathbb{F}_2, -)$.

7. Examples: $\mathrm{Ext}_{\mathcal{E}(1)^\vee}(\mathbb{F}_2, B_0(1))$ and $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, B_0(1))$

Let's do two examples with the algebraic Atiyah-Hirzebruch spectral sequence to see how this all fits together.

7.1. $\mathbf{E} = \mathbf{bu}$.

We are going to study the algebraic Atiyah-Hirzebruch spectral sequence

$$E_1 = \bigoplus_{0 \leq i \leq k} \mathrm{Ext}_{\mathcal{E}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{gr}_i^{\mathrm{cell}} B_0(1) \implies \mathrm{Ext}_{\mathcal{E}(1)^\vee}(\mathbb{F}_2, B_0(1)).$$

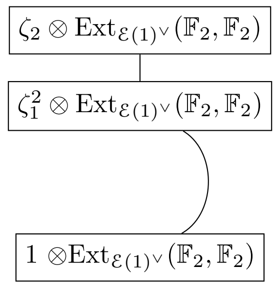
The cellular filtration on $B_0(1)$ is simple:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{F}_0 B_0(1) \arrow[r] \arrow[d] & \mathrm{F}_1 B_0(1) \arrow[r] \arrow[d] & \mathrm{F}_2 B_0(1) \arrow[r] \arrow[d] & \mathrm{F}_3 B_0(1) = B_0(1) \arrow[d] \\ & \mathbb{F}_2\{1\} & 0 & \mathbb{F}_2\{\zeta_1^2\} & \mathbb{F}_2\{\zeta_2\} \end{tikzcd}$$

In the cell diagram for $B_0(1)$, we have a single cell in degrees 0, 2, and 3, giving us the associated graded structure depicted. This lets us rewrite the E_1-page.

$$E_1 = \mathrm{Ext}_{\mathcal{E}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathbb{F}_2\{1, \zeta_1^2, \zeta_2\}.$$

Remember that the differentials are given by the attaching maps in the Brown-Gitler comodule. Heurestically, we should have the following picture in mind:



Each of the cells in $B_0(1)$ has a copy of $\mathrm{Ext}_{\mathcal{E}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2)$ attached to it. We can see where there will be differentials in the spectral sequence directly from this picture. There is a d_1-differential between Atiyah-Hirzebruch filtrations 3 and 2 given by the cells in degrees 3 and 2. The comodule structure map realizes this differential as multiplication by v_0 in the spectral sequence.

To be a little more precise, for a class

$$x \in \mathrm{Ext}_{\mathcal{E}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[v_0, v_1],$$

let $x[n]$ denote the copy of this class attached to the cell in Atiyah-Hirzebruch filtration n (if we had multiple classes in each degree, then this would be poor notation, but it works for this case). Then the differential is given by

$$d_1(x[3]) = h_0 x[2].$$

Each differential is linear over $\mathrm{Ext}_{\mathcal{E}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2)$, which determines the E_1-page. Below, we give the E_1-page and suppress Atiyah-Hirzebruch filtration. Again, this gives a copy of $\mathrm{Ext}_{\mathcal{E}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2)$ for each associated graded coming from the cellular filtration on $B_0(1)$. This gives a copy at stem 0 for the cell $\{1\}$, a copy at stem 2 for the cell $\{\zeta_1^2\}$, and a copy at stem 3 for the cell $\{\zeta_2\}$.

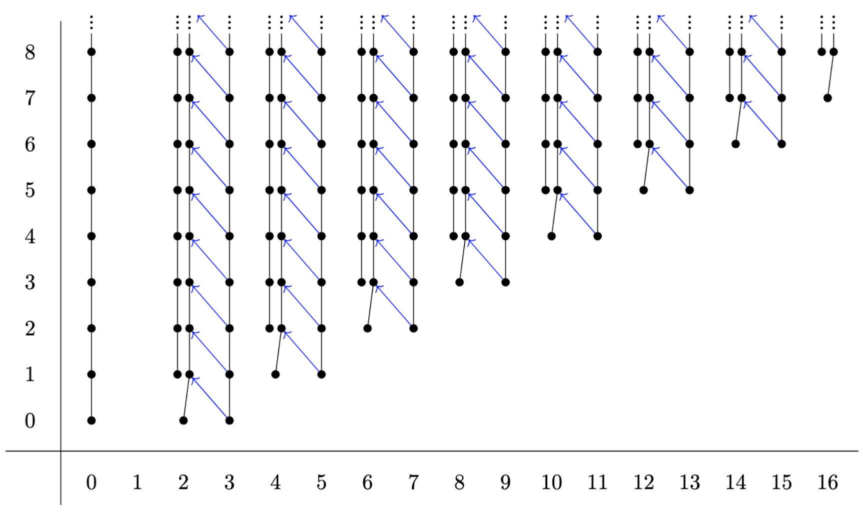


FIGURE 3. The E_1-page of the bu-**aAHSS**$(B_0(1))$

For degree reasons, there are no other possible differentials, hence $E_2 = E_\infty$. We also can see this in the cobar complex: the attaching map giving the d_2-differential is given by h_1, which is represented by $[\zeta_1^2]$. This class is absent from the cobar complex. Notice that the E_∞-page is populated by all of the terms from Atiyah-Hirzebruch filtration 0, the bases of the v_0-towers from Atiyah-Hirzebruch filtration 2, and nothing fro Atiyah-Hirzebruch filtration 3. To solve hidden extensions, we can again look at the cobar complex. This is a little more subtle, but when we work through the details (I'm not going to here) we get that there is a hidden h_0 extension:

$$h_0 \cdot v_1^i[2] = v_1^{i+1}$$

for all $i \geq 0$.

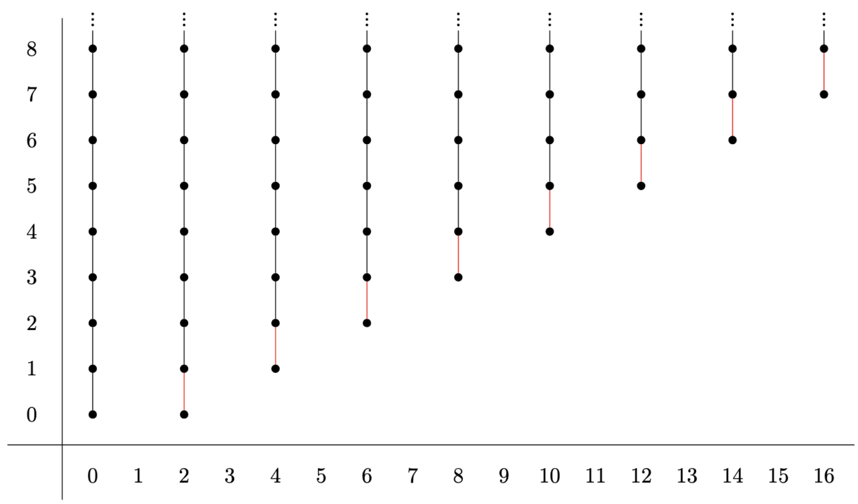


FIGURE 4. The E_∞-page pf the bu-**aAHSS**$(B_0(1))$ with hidden extensions in red.

7.2. $\mathbf{E} = \mathbf{bo}$. We are going to study the algebraic Atiyah-Hirzebruch spectral sequence

$$E_1 = \bigoplus_{0\leq i\leq k} \mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{gr}_i^{\mathrm{cell}} B_0(1) \implies \mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, B_0(1)).$$

The cellular filtration on $B_0(1)$ is the same as the previous example, and so we can rewrite the E_1-page:

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathbb{F}_2\{1, \zeta_1^2, \zeta_2\}.$$

The differentials are determined in the same way as before, but we will see that this spectral sequence doesn't collapse as quickly. The d_1-differential is given by multiplication by h_0 on classes in Atiyah-Hirzebruch filtration 3. We give charts below.

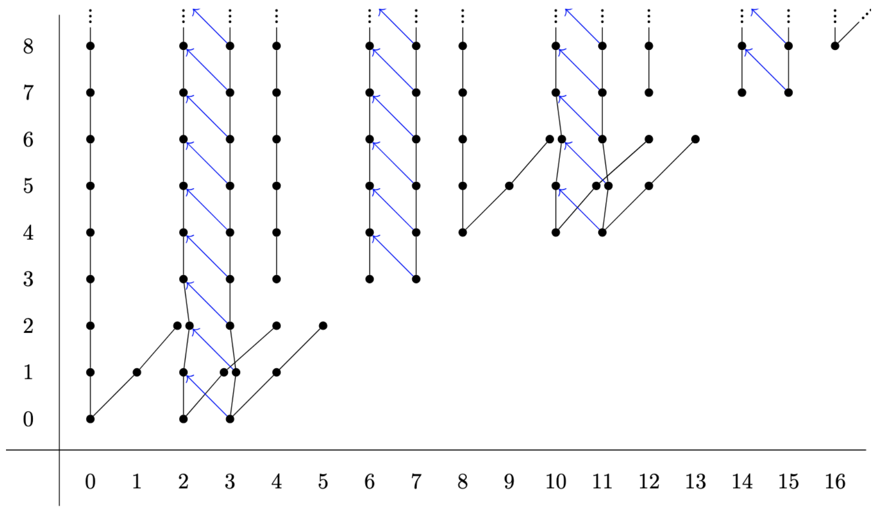


FIGURE 5. The E_1-page of the bo-**aAHSS**$(B_0(1))$

The d_2-differential is given by the h_1-attaching map between Atiyah-Hirzebruch filtrations 2 and 0. Notice that unlike in the previous example, this is not trivial! You can see this from the cobar complex, or by looking at the charts, but the real reason is that h_1 detects η in the sphere, and η is in the Hurewicz image of $\mathbb{S} \to \mathrm{bo}$.

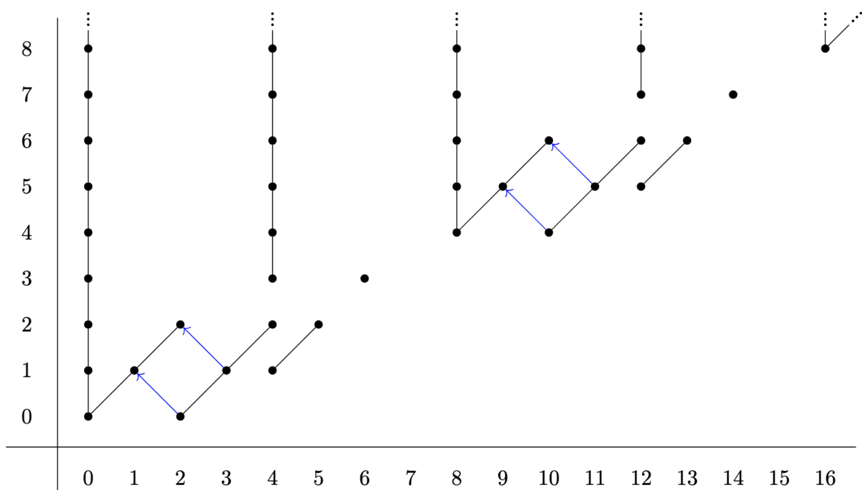


FIGURE 6. The E_2-page of the bo-**aAHSS**$(B_0(1))$

For degree reasons, there are no other differentials. Hidden extensions can be solved using the cobar complex as mentioned above. However, there's actually a really slick trick you can use here to get the hidden extensions. Mahowald and Milgram showed that there is an equivalence of spectra [MM76]:

$$\mathrm{bsp} \simeq \mathrm{bo} \otimes \mathrm{H}\mathbb{Z}_1,$$

where bsp is the connective cover of $\Sigma^4\mathrm{KO}$. The E_2-page of the Adams spectral sequence for bsp, whose homotopy groups we know by Bott periodicity, allows for a change of rings isomorphism:

$$E_2 = \mathrm{Ext}_{\mathcal{A}^\vee}(\mathbb{F}_2, \mathrm{H}_*(\mathrm{bo}) \otimes \mathrm{H}_*(\mathrm{H}\mathbb{Z}_k)) \cong \mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2, B_0(1)).$$

In particular, the group that we just calculated by the algebraic Atiyah-Hirzebruch is actually the E_2-page of the Adams spectral sequence for bsp. Knowing the structure of π_*bsp and observing that there can be no differentials gives the hidden extensions.

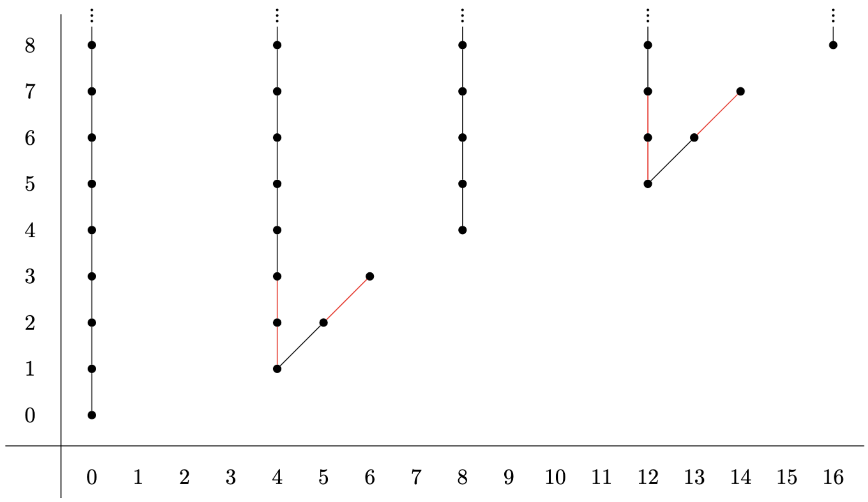


FIGURE 7. The E_∞-page of the bo-$\mathbf{aAHSS}(B_0(1))$ with hidden extensions in red.

A lot of the charts given in this example can be found in [Mah81], and he is essentially (maybe literally) doing the same process I just described.

8. The section where I wax poetically

For some, what I just outlined is a fun game and useful tool for computation. While we only computed the first part of the E_2-page for the ring of cooperations, and only in a few select cases, one can extend to understand the entire E_2-page by using either another series of algebraic Atiyah-Hirzebruch spectral sequences or some clever short exact sequences of comodules, or whatever. In the cases I described, the spectral sequences actually collapse (modulo some irrelevant torsion). You can see these methods in practice in more than just stable homotopy theory [Mah81, BOSS19], but also in equivariant homotopy theory [LPT25] and motivic homotopy theory [CQ21, Mor25].

Maybe just to sum everything up, here's a very general way to compute $[\mathbf{1}, \mathrm{E}\otimes\mathrm{E}]$ in an arbitrary stable, presentably symmetric monoidal ∞-category.

(1) Find a nice object H (like a flat, connective ring spectrum with a Künneth isomorphism whose H-based Adams spectral sequence converges to something not for away from the cooperations) where $\mathrm{H}_*(\mathrm{E}) = [\mathbf{1}, \mathrm{H}\otimes\mathrm{E}]$ is a particularly nice comodule over $[\mathbf{1}, \mathrm{H}\otimes\mathrm{H}]$.
(2) Use the H-based Adams spectral sequence to compute the cooperations. Use the Künneth isomorphism and change-of-rings isomorphism to rewrite the E_2-page as something like
$$\mathrm{Ext}_{[\mathbf{1},\mathrm{H}\otimes\mathrm{H}]}([\mathbf{1},\mathrm{H}],[\mathbf{1},\mathrm{E}\otimes\mathrm{E}]) \cong \mathrm{Ext}_{\mathrm{B}}([\mathbf{1},\mathrm{H}],[\mathbf{1},\mathrm{E}]).$$
(3) Hope and pray for (or sit down and construct) a family of finite comodules M_n which filter B and give rise to a splitting of $[\mathbf{1}, \mathrm{E}]$ as a comodule over B:
$$[\mathbf{1},\mathrm{E}] \cong \bigoplus_{k\geq 0} \Sigma^{n_k}\mathrm{M}_k.$$

Do this in a way so that $\mathrm{M}_0 = [\mathbf{1}, \mathrm{H}]$. This will lead to a splitting of the E_2-page of the Adams spectral sequence:

$$\mathrm{Ext}_{\mathrm{B}}([\mathbf{1}, \mathrm{H}], [\mathbf{1}, \mathrm{E}]) \cong \bigoplus_{k \geq 0} \Sigma^{n_k} \mathrm{Ext}_{\mathrm{B}}([\mathbf{1}, \mathrm{H}], \mathrm{M}_k)$$

(4) Use a known computation of (or compute for yourself) the group

$$\mathrm{Ext}_{\mathrm{B}}([\mathbf{1}, \mathrm{H}], [\mathbf{1}, \mathrm{H}]),$$

and filter M_1 in some type of cellular way (if things are graded, filter be degree). Apply $\mathrm{Ext}_{\mathrm{B}}([\mathbf{1}, \mathrm{H}], -)$ to this filtration to get an algebraic Atiyah-Hirzebruch spectral sequence

$$E_1 = \bigoplus_{0 \leq i \leq n} \mathrm{Ext}_{\mathrm{B}}([\mathbf{1}, \mathrm{H}], [\mathbf{1}, \mathrm{H}]) \otimes \mathrm{gr}_i \mathrm{M}_1 \implies \mathrm{Ext}_{\mathrm{B}}([\mathbf{1}, \mathrm{H}], \mathrm{M}_1);$$

if the comodule M_1 is finite, then this is terminates at a finite stage. Hopefully even it's easy!

(5) Use some short exact sequences relating the family of comodules M_i, possibly with another related family of comodules M_i', to induct up to a computation of

$$\mathrm{Ext}_{\mathrm{B}}([\mathbf{1}, \mathrm{H}], \mathrm{M}_k),$$

hence describing the E_2-page of the Adams spectral sequence computing the cooperations.

Once you've made it to the E_2-page of the Adams spectral sequence, all bets are up in the air on how to compute the differentials. Maybe you can argue that there are some elements which are a particular type of torsion, and you want to ignore that particular type of torsion, so the spectral sequence collapses for degree reasons. Maybe you can do something else.

We mentioned that this philospy extends in the case of spectra to describing the E_1-page of the E-based Adams spectral sequence, line by line. Let's look at the example of $\mathrm{E} = \mathrm{bo}$ again. The n-line of the bo-resolution takes the form

$$E_1^{*,n} = \pi_{*+n}(\mathrm{bo} \otimes \overline{\mathrm{bo}}^{\otimes n}).$$

We may compute the n-line by another Adams spectral sequence. The Künneth isomorphism implies that

$$\mathrm{H}_*(\mathrm{bo} \otimes \overline{\mathrm{bo}}^{\otimes n}) \cong \mathrm{H}_*(\mathrm{bo}) \otimes \mathrm{H}_*(\overline{\mathrm{bo}})^{\otimes n}.$$

After using the change of rings isomorphism on $\mathrm{H}_*(\mathrm{bo})$, the $\mathbf{ASS}(\mathrm{bo} \otimes \overline{\mathrm{bo}}^{\otimes n})$ takes the form

$$E_2 = \bigoplus_{I \in \mathcal{I}_n} \Sigma^{4|I|} \mathrm{Ext}_{\mathcal{A}(1)^\vee}(B_0(I)) \implies \pi_{*+n}(\mathrm{bo} \otimes \overline{\mathrm{bo}}^{\otimes n}),$$

where $\mathcal{I}_n = \{I = (i_1, \ldots, i_n) : i_j \geq 0 \text{ for all } 1 \leq j \leq n\}$, and if $I = (i_1, \ldots, i_n)$, then $|I| = i_1 + \cdots + i_n$ and $B_0(I) = B_0(i_1) \otimes \cdots \otimes B_0(i_n)$. Once one has determined the structure of the groups $\mathrm{Ext}_{\mathcal{A}(1)^\vee}(B_0(k))$ through the process of computing the ring of cooperations, one can study this spectral sequence in detail to compute the E_1-page of the bo-resolution line by line.

Remark 8.1. This process of computation only describes the structure of the ring of cooperations $\pi_*(\mathrm{E} \otimes \mathrm{E})$ as a module over $\pi_*\mathrm{E}$; we are saying practically nothing about the ring structure, apart from the inherent ring structure on the submodule $\mathrm{Ext}_{\pi_*(\mathrm{E}\otimes\mathrm{E})}(\pi_*\mathrm{E}, \pi_*\mathrm{E})$. However, as far as the E_1-page of the E-**ASS** goes, this is not important. While the entire E_1-page has an algebra structure, the 1-line is given by the algebra $\pi_*(\mathrm{E} \otimes \overline{\mathrm{E}})$, which is not quite equivalent to the ring of cooperations. In fact, the spectrum $\mathrm{E} \otimes \overline{E}$ is *not* a ring! This generalizes: the n-line of the **E-ASS** is not a ring in its own right for $n \geq 1$. This style of computation of the n-line give us the module structure over the 0-line. The multiplicative structure of the E_1-page is (probably) given by some type of concatenation.

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