Cooperations in motivic homotopy theory

Jackson Morris (University of Washington)

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Motivic Homotopy Theory

"Homotopy theory of schemes"

$\mathrm{Sm}_F \longrightarrow \mathrm{Spc}_F = \mathrm{L}_{\mathbb{A}^1,\mathrm{Nis}}\mathrm{Fun}(\mathrm{Sm}_F^{\mathrm{op}}, \mathcal{S})$

Presheaves of spaces satisfying

Nisnevich descent and $\mathbb{A}^1$- invariance

It's not really that bad!

Yoneda embedding

$\mathrm{Sm}_F \hookrightarrow \mathrm{Spc}_F$

Constant functor

$\mathcal{S} \to \mathrm{Spc}_F$

$\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$

Bigraded homotopy groups

$S^{s,w} = (S^1)^{\wedge s-w} \wedge (\mathbb{G}_m)^{\wedge w}$

$\pi_{s,w}X = [S^{s,w}, X]_{\mathbb{A}^1}$

invert $\wedge \mathbb{P}^1$ $\longrightarrow$

Stable motivic homotopy theory

$\mathrm{SH}(F)$

Stable symmetric monoidal category

$\Sigma^\infty_{\mathbb{P}^1}\mathrm{Spec}(F) := \mathbb{S}$ motivic sphere spectrum

Goal: understand $\pi^F_{**}\mathbb{S}$

What do we know?

$$\pi^F_{**}\mathbb{S}$$

<u>Morel (2000s):</u> $\pi^F_{n,n}\mathbb{S} \cong K^{MW}_{-n}(F)$

"No Serre's finiteness"

<u>RSØ (2010s):</u> $\pi^F_{n+1,n}\mathbb{S}$ and $\pi^F_{n+2,n}\mathbb{S}$

Arithmetic alert!

<u>BBX (2025):</u> $(\pi^F_{s,w}\mathbb{S}^{\wedge}_{\ell,\eta}) \cong (\pi^{syn}_{s,w} \otimes K^{MW}(F))^{\wedge}_{\ell}$

Topology alert!

<u>Levine (2010s):</u> $\pi^{\mathbb{C}}_{s,0}\mathbb{S} \cong \pi_s\mathbb{S}$

How can we organize $\pi^F_{**}\mathbb{S}$?

<u>WØ (2010s):</u> $\pi^{\mathbb{F}_q}_{s,0}\mathbb{S}[p^{-1}] \cong \pi_s\mathbb{S}[p^{-1}]$

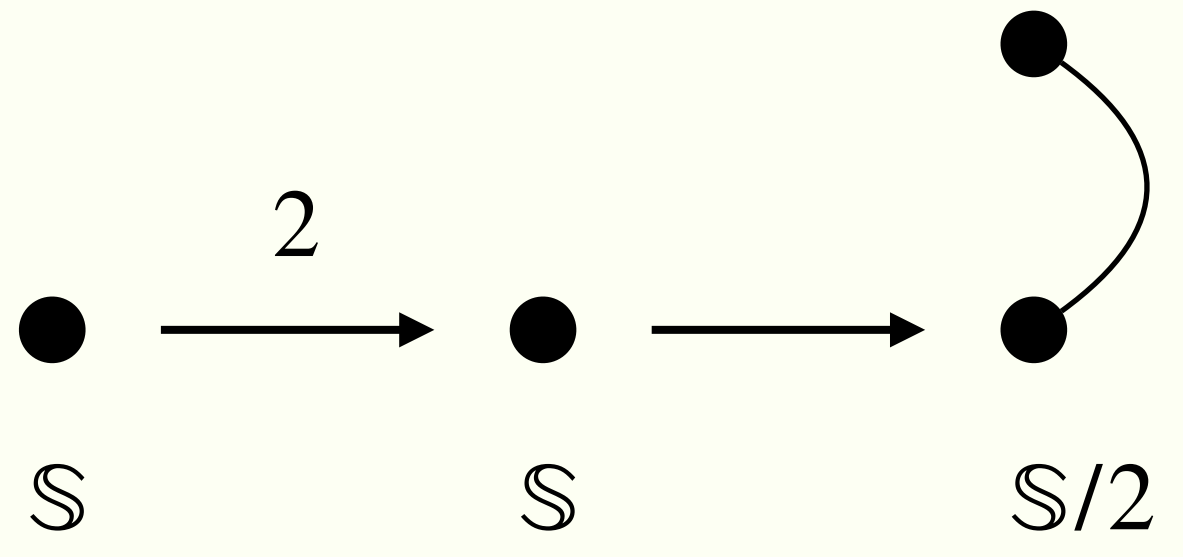
The motivic Hopf map $\eta \in \pi^F_{1,1}\mathbb{S}$ is non-nilpotent.

Chromatic homotopy theory!

"No Nishida's nilpotence"

Periodic elements in $\pi^F_{**}\mathbb{S}$

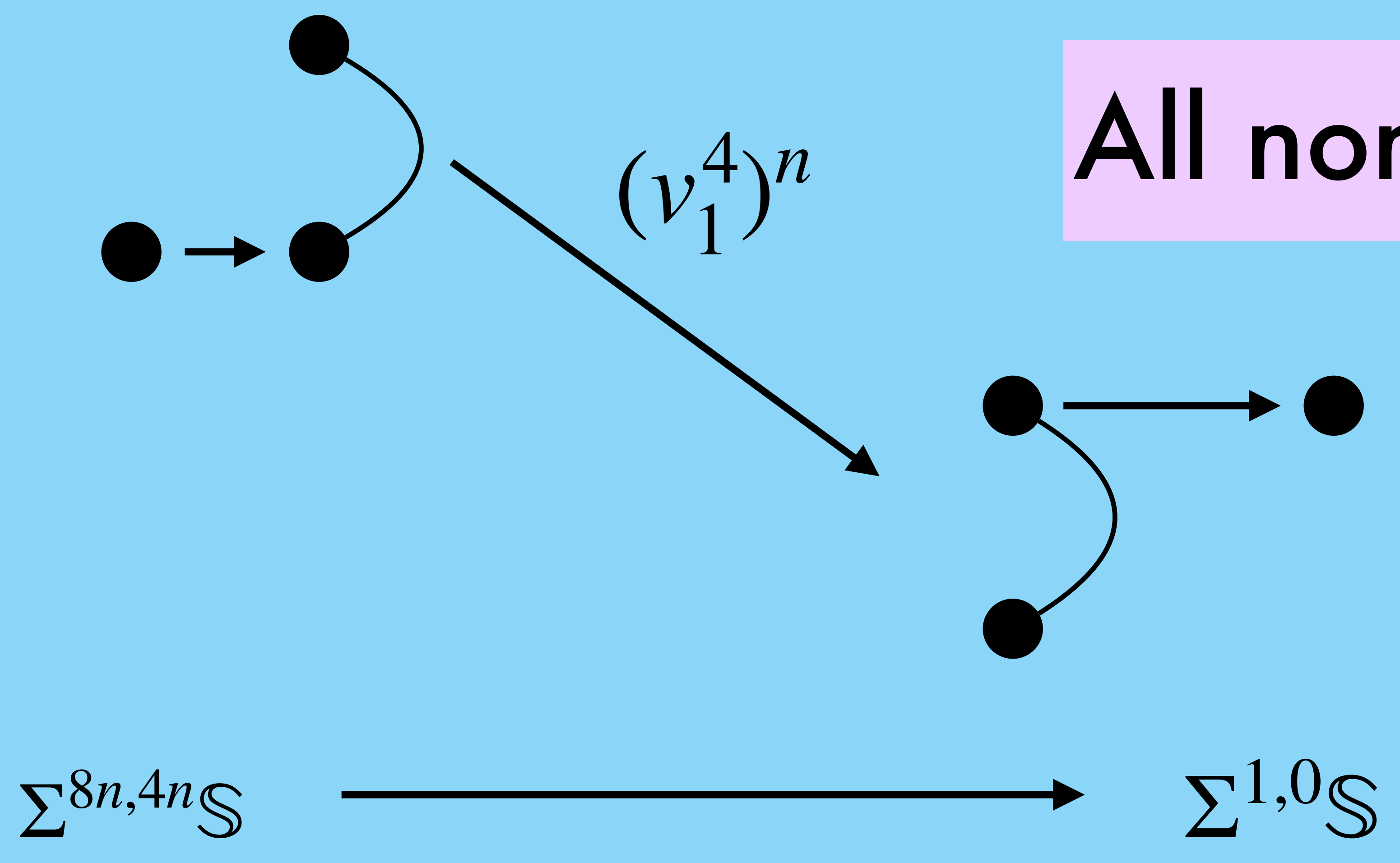
Take the element $2 \in \mathbb{Z} \subset \pi^F_{0,0}\mathbb{S}$



There is a non-nilpotent map

$$v_1^4 : \Sigma^{8,4}\mathbb{S}/2 \to \mathbb{S}/2$$

We can use this to find elements in $\pi^F_{**}\mathbb{S}$!



All nontrivial elements in $\pi^F_{8n-1,4n}\mathbb{S}$

Refined Goal:

understand the v_1-periodicity

of $\pi^F_{**}\mathbb{S}$

The Adams spectral sequence

$\pi^F_{**}(E \otimes E)$

Ring of

cooperations

Machine that turns hard problems into annoying problems

$$E_1^{s,f,w} = \pi^F_{s+f,w}(E \otimes E^{\otimes f}) \implies \pi^F_{s,w}\mathbb{S}$$

$$\Uparrow$$

$$E_2 = Ext_{A^\vee}(\mathbb{M}_p, H_{**}(E \otimes E^{\otimes f}))$$

Different E see different parts of $\pi^F_{**}\mathbb{S}$

Bootstrap up from $\pi^F_{**}(E \otimes E)$

Choose E that sees v_1-periodic part of $\pi^F_{**}\mathbb{S}$!

To compute the E_1-page of the E-based Adams spectral sequence

Two good choices of E:

kq - Hermitian K-theory

$BPGL\langle 1\rangle$- Truncated Brown-Peterson

Theorem (M. 2025)

Computed the ring of cooperations

$$\pi^F_{**}(kq \otimes kq)$$

for $F \in \{\mathbb{R}, \mathbb{F}_q\}$ at the prime 2.

Explicit description of E_1-page of kq-based Adams SS.

Theorem (M.-Petersen-Tatum 2025)

Computed the ring of cooperations

$$\pi^F_{**}(BPGL\langle 1\rangle \otimes BPGL\langle 1\rangle)$$

for $F \in \{\mathbb{C}, \mathbb{R}, \mathbb{F}_q\}$ at all primes.

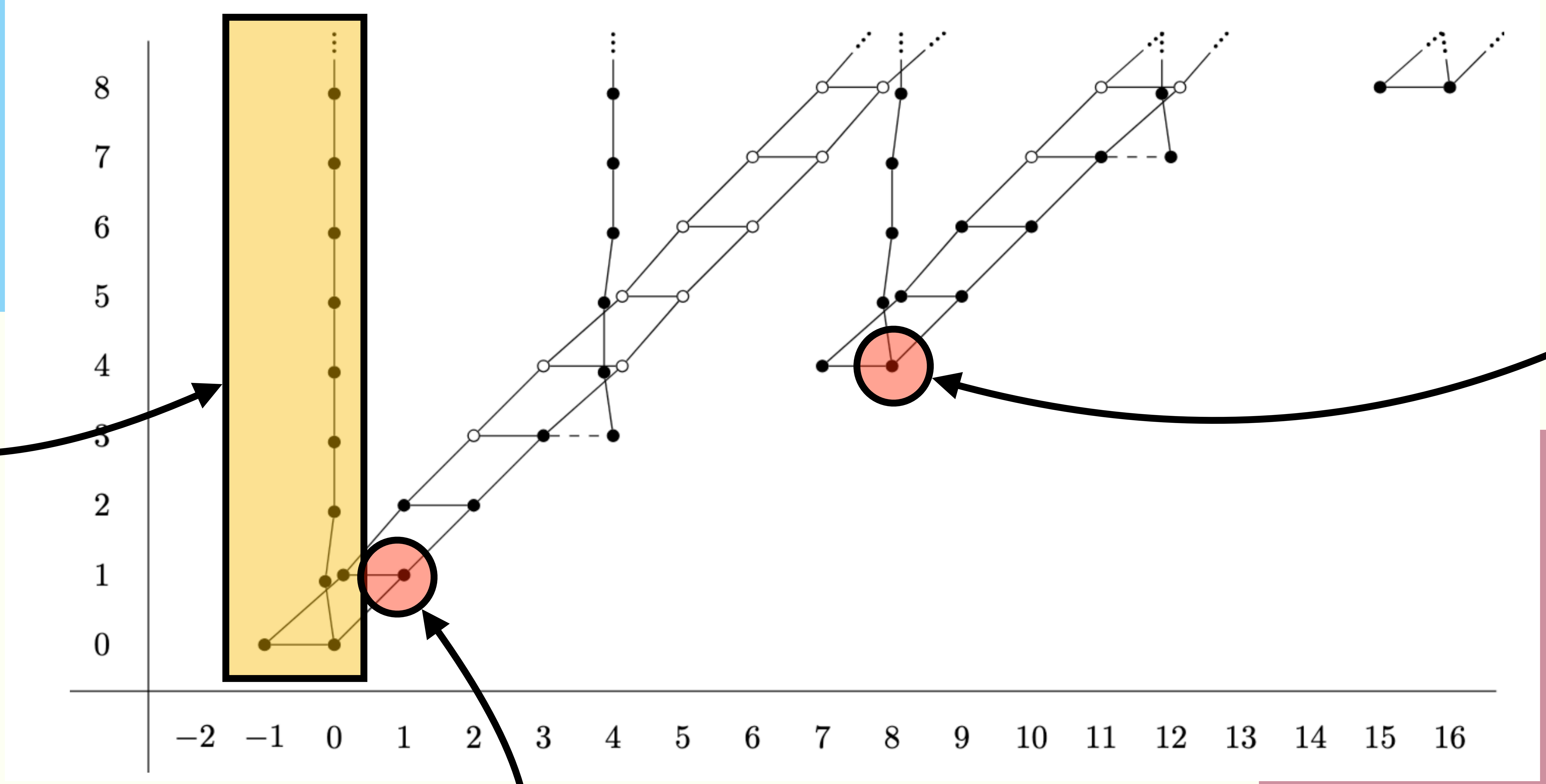
Explicit description of E_1-page of $BPGL\langle 1\rangle$-based Adams SS.

Picture Time!

Look at kq over $\mathbb{F}_3$

The homotopy groups $\pi^{\mathbb{F}_3}_{**}(kq)$

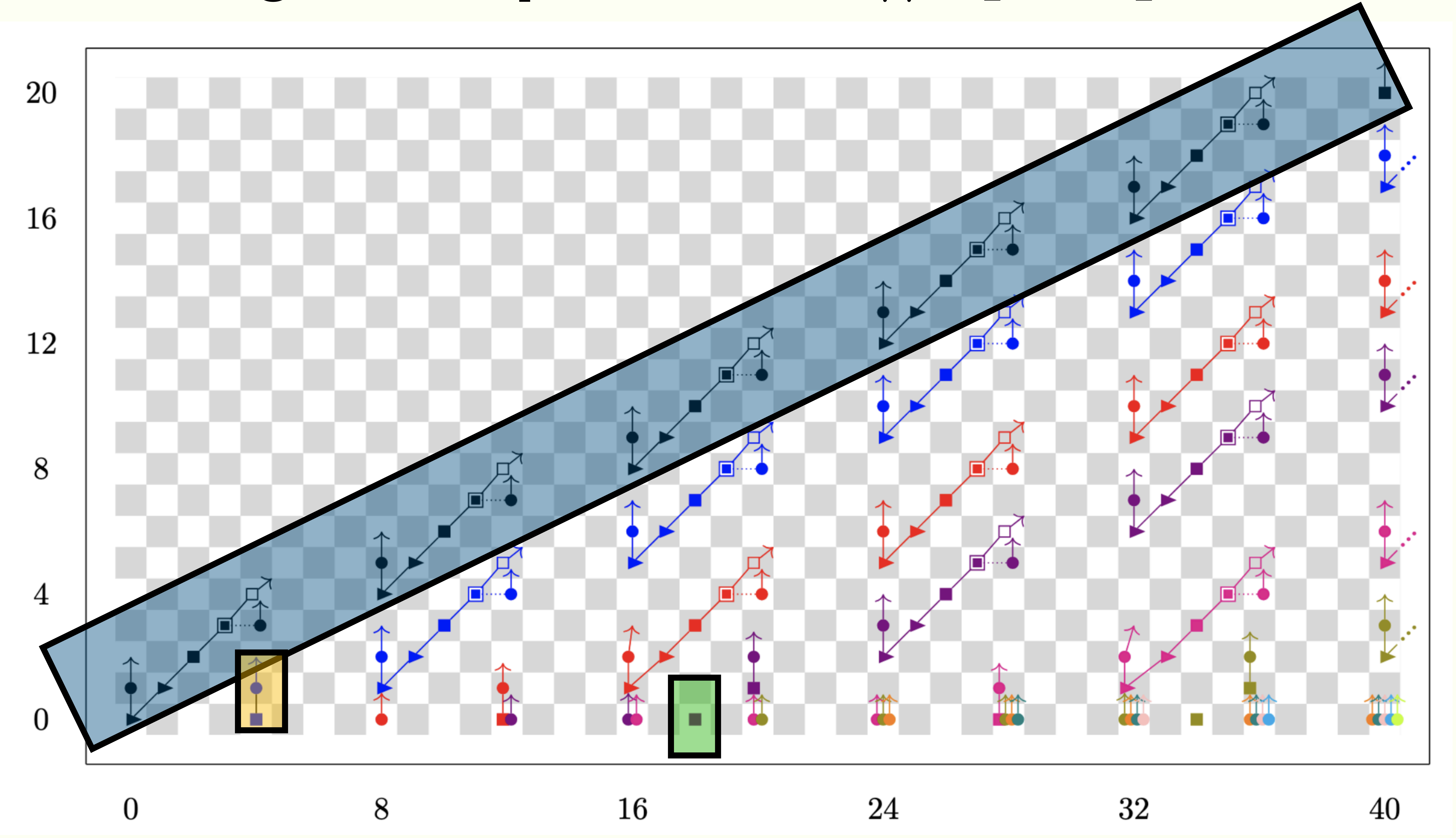
$\pi^{\mathbb{F}_3}_{**}H\mathbb{Z}$



v_1^4

η

The ring of cooperations $\pi^{\mathbb{F}_3}_{**}(kq \otimes kq)$



Each color is a different summand

Each summand consists of:

$\pi^{\mathbb{F}_3}_{**}(kq)$

$\pi^{\mathbb{F}_3}_{**}H\mathbb{Z}$'s

(and variants)

$\pi^{\mathbb{F}_3}_{**}H\mathbb{F}_2$

(and variants)

$\pi^{\mathbb{F}_3}_{**}(kq)$-module structure

This is computable/understandable!

Each summand is built upon $\pi^{\mathbb{F}_3}_{**}(kq)$

The E_1-page of the E-Adams SS (for $E \in \{kq, BPGL\langle 1\rangle\}$)

These are the algebraic summands from before!

Just bookkeeping.

$$E_2^{s,f,w} = Ext_{A^\vee}^{s,f,w}(\mathbb{M}_p, H_{**}(E \otimes E^{\otimes n})) \implies \pi^F_{s,w}(E \otimes E^{\otimes n})$$

$$\cong \bigoplus_{K \in I_E} \Sigma^K Ext_{A^\vee}^{s,f,w}(\mathbb{M}_p, B_0(K))$$

Finite free $H\mathbb{F}_p$-module

For $BPGL\langle 1\rangle$, we can say more.

Thm (MPT 2025)

There is an equivalence

$$BPGL\langle 1\rangle \otimes BPGL\langle 1\rangle \simeq \bigoplus_{k \geq 0} \Sigma^{2k,k} BPGL\langle 1\rangle^{\nu_p(k!)} \oplus V$$

$\nu_p(k!)^{th}$ Adams cover

Thm (M, MPT 2025)

Computed the E_1-page of the E-Adams SS as a module over $\pi^F_{**}E$.

What's next?

Determine the v_1-periodicity of $\pi^F_{**}\mathbb{S}$

Run the kq and $BPGL\langle 1\rangle$-based Adams SS

Extend results to other base schemes

Pullback square of Bachmann-Østvaer

$$\begin{array}{ccc} (E\otimes E)(\mathbb{Z}[1/2]) & \longrightarrow & (E\otimes E)(\mathbb{R}) \\ \downarrow & \lrcorner & \downarrow \\ (E\otimes E)(\mathbb{F}_3) & \longrightarrow & (E\otimes E)(\mathbb{C}) \end{array}$$

Determine exotic periodicity in $\pi^F_{**}\mathbb{S}$

Use other E-based Adams SS

C_2-equivariant periodicity

$\mathbb{R}$-motivic computations are the "positive cone" of analogous C_2-equivariant computations

Motivic height 1 telescope conjecture

This work is a motivic analogue of Mahowald's seminal work on bo-resolutions

THANK YOU!

References

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