HIGHER REAL K-THEORIES, REDSHIFT, AND BLUESHIFT

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Abstract. An expanded set of notes for a talk in the UW DUBTOP seminar.

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Part 1. Higher Real K-theories

1. Introduction

A central goal in homotopy theory is to understand $\pi_*(\mathbb{S})$, the stable homotopy groups of spheres. Let's make some first observations.

- By Serre's finiteness theorem [Ser53], for $k > 0$ we have that $\pi_k(\mathbb{S})$ is a finite abelian group. Thus,

$$\pi_k(\mathbb{S}) = \begin{cases} \mathbb{Z} & k = 0 \\ \text{finite abelian} & k > 0 \\ 0 & k < 0. \end{cases}$$

- The sphere spectrum is the unit for the symmetric monoidal structure on spectra, and so the stable homotopy groups of spheres assemble into a graded ring. By Nishida's theorem [Nis73], for $k > 0$ and $\alpha \in \pi_k(\mathbb{S})$, there is some $N > 0$ such that $\alpha^N = 0$. In other words, the multiplicative structure of the graded commutative ring $\pi_*(\mathbb{S})$ is not the easiest to get our hands on.

Since the multiplicative structure is complicated, let's instead concentrate on just the groups appearing in $\pi_*(\mathbb{S})$.

2. Arithmetic fracture

One thing we can do is break this problem into smaller, more computable pieces, which assemble to our end goal. For any spectrum X, there is a homotopy pullback square which we will call the *arithmetic fracture square*

$$\begin{tikzcd} \mathrm{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{X}_{\mathbb{Q}} \arrow[d] \\ \prod_p \mathrm{X}_p \arrow[r] & \left(\prod_p \mathrm{X}_p\right)_{\mathbb{Q}} \end{tikzcd} \tag{2.1}$$

Here X_p denotes the *p-completion* of X and $\mathrm{X}_{\mathbb{Q}}$ denotes the *rationalization* of X.

Let's think about this for $\mathrm{X} = \mathbb{S}$. Since $\mathbb{Q} \otimes A \simeq 0$ for any finite abelian group A, Serre's finiteness theorem implies that

$$\pi_k(\mathbb{S}_{\mathbb{Q}}) \cong \begin{cases} \mathbb{Q} & k = 0 \\ 0 & \text{else.} \end{cases}$$

The Postnikov section $\mathbb{S}_{\mathbb{Q}} \to \mathbb{Q}$ then is an equivalence, so that $\mathbb{S}_{\mathbb{Q}} \simeq \mathbb{Q}$. Similarly, the bottom righthand corner of the diagram is a product of $\mathbb{Q}_p$'s. So, we only have to understand the p-complete sphere $\mathbb{S}_p$!

Before diving into this task, let's see what happens in algebra first. Suppose we wanted to understand the effect that p has on the p-adic integers $\mathbb{Z}_p$. By "effect", I really mean one of two things:

- If $x \in \mathbb{Z}_p$, is x p-power torsion?
- If $x \in \mathbb{Z}_p$, is x p-power torsionfree?

In the latter case, we will say that x is *p-periodic*. A simple procedure we can do in algebra is invert p. This is a localization, and the image of the localization map will be those elements which are p-periodic. Since $p^{-1}\mathbb{Z}_p \simeq \mathbb{Q}_p$, this localization map takes the form

$$\mathbb{Z}_p \to \mathbb{Q}_p.$$

Notice that $\mathbb{Q}_p$ is a field, and hence there is not really anything interesting going on (there are no nontrivial ideals). Moreover, we can take the cofiber of this map, and it turns out that $\mathbb{Q}_p/\mathbb{Z}_p \cong \mathrm{colim}_n(\mathbb{Z}/p^n)$. Hence, there is a cofiber sequence for understanding the effect of p on $\mathbb{Z}_p$ which takes the form

$$\mathbb{Z}_p \to \mathbb{Q}_p \to \mathrm{colim}_n(\mathbb{Z}/p^n).$$

Notice that $\mathbb{Z}/p$ is also a field, so its algebra is quite simple. This implies that there is no higher periodic phenomena in the p-torsion of $\mathbb{Z}/p$, and gives us a very good understanding of the algebra surrounding $\mathbb{Z}_p$.

Let's switch back to $\mathbb{S}_p$. Since $\pi_0(\mathbb{S}_p) \cong \mathbb{Z}_p$, we can again ask which elements in $\pi_*(\mathbb{S}_p)$ are p-torsion or p-periodic. By Serre's finiteness theorem there can be no elements in $\pi_k(\mathbb{S}_p)$ for $k > 0$ which are p-periodic. So, just as in algebra, the localization $p^{-1}\mathbb{S}_p$ is equivalent to $\mathbb{Q}_p$. We can repackage this as a cofiber sequence

$$\mathbb{S}_p \to \mathbb{Q}_p \to \mathrm{colim}_n(\mathbb{S}/p^n).$$

Here is where things get interesting: the spectrum $\mathbb{S}/p$ is *not* a field object in Sp! This observation is basically the birth of "higher algebra" or "chromatic homotopy theory" or whatever.

Recall that we can think of $p \in \pi_0(\mathbb{S}_p)$ as a self-map $p : \mathbb{S}_p \to \mathbb{S}_p$ of degree 0. Since $p \in \mathbb{Z}_p$ is non-nilpotent, this allows us to think of the localization as inverting this self-map:

$$\mathbb{Q}_p = p^{-1}\mathbb{S}_p := \mathrm{colim}(\mathbb{S}_p \xrightarrow{p} \mathbb{S}_p \xrightarrow{p} \cdots).$$

The key finding of Adams is that $\mathbb{S}_p$ also has a non-nilpotent self-map [Ada66]. This map often goes by the name v_1 and takes the form

$$v_1 : (\mathbb{S}_p/p)[d] \to \mathbb{S}_p/p.$$

Here, I am using $[d] := \Sigma^d$. For $p = 2$, $d = 8$ and this map ought to be named v_1^4 for reasons to be explained later. For p odd, $d = 2(p-1)$. Since this map is non-nilpotent, we may invert it! This allows us to talk about v_1-periodic elements in $\pi_*(\mathbb{S}_p/p)$. Form the localization

$$v_1^{-1}(\mathbb{S}_p/p) = \mathrm{colim}(\mathbb{S}_p/p \xrightarrow{v_1} (\mathbb{S}_p/p)[-d] \xrightarrow{v_1} \cdots).$$

The v_1-periodic elements in $\mathbb{S}_p/p$ are those which have nontrivial image under the localization map

$$\mathbb{S}_p/p \to v_1^{-1}(\mathbb{S}_p/p).$$

This further divides the p-complete sphere into those elements which, after modding out by p, are v_1-torsion or v_1-periodic. By precomposing with the quotient map and ask about v_1-periodic elements in $\mathbb{S}_p$, i.e. those which have nontrivial image under the composite

$$\mathbb{S}_p \to \mathbb{S}_p/p \to v_1^{-1}(\mathbb{S}_p/p).$$

So, what are the v_1-periodic elements in the sphere? We'll get back to this.

3. Chromatic fracture

As one may guess, this pattern repeats indefinitely. There is an operator v_2 which acts on the v_1-periodic part of the sphere, hence we may ask which elements are v_2-torsion or v_2-periodic, and on and on. Let's make this a little more formal.

Remark 3.1. I am going to be quite brief with my discussions, and I will not talk about formal geometry. For some deeper background on chromatic homotopy theory, see [Rav86; Pet19; BB20; COCTALOS; QCohMfg; LurCHT; RogCHT; GuiCHT; PstCHT].

Remark 3.2. For the rest of this article, everything will be implicitly p-complete. I will omit p-completion from notation.

Let's think in terms of localizations. Let E be some spectrum. We say that X is *E-acyclic* if $\mathrm{X} \otimes \mathrm{E} \simeq 0$. A map of spectra $f : \mathrm{X} \to \mathrm{Y}$ is an *E-equivalence* if $f : \mathrm{X} \otimes \mathrm{E} \to \mathrm{Y} \otimes \mathrm{E}$ is an equivalence. Finally, X is *E-local spectra* if X sees E-equivalences as equivalences. This means that the following equivalent statements are true:

- if $f : \mathrm{Y} \to \mathrm{Z}$ is an E-equivalence, then
$$\mathrm{map}(f, \mathrm{X}) : \mathrm{map}(\mathrm{Z}, \mathrm{X}) \to \mathrm{map}(\mathrm{Y}, \mathrm{X})$$
is an equivalence;
- if Y is E-acyclic, then $\mathrm{map}(\mathrm{Y}, \mathrm{X}) \simeq *$.

The collection Sp_{E} of E-local spectra is a full subcategory which satisfies the right technical conditions so that the inclusion $\mathrm{Sp}_{\mathrm{E}} \subseteq \mathrm{Sp}$ admits a right adjoint, known as the *Bousfield localization* $L_{\mathrm{E}}(-) : \mathrm{Sp} \to \mathrm{Sp}_{\mathrm{E}}$. We often denote $\mathrm{X}_{\mathrm{E}} := L_{\mathrm{E}}(\mathrm{X})$.

The first localization we encountered was rationalization $L_{\mathbb{Q}}(-) : \mathrm{Sp} \to \mathrm{Sp}_{\mathbb{Q}}$. It basically follows from Serre's finiteness again (what a banger!) that there is an equivalence of categories $\mathrm{Sp}_{\mathbb{Q}} \simeq \mathrm{Ch}(\mathbb{Q})$. Actually, this localization is *smashing*, meaning that $\mathrm{X}_{\mathbb{Q}} \simeq \mathrm{X} \otimes \mathbb{Q}$.

The second localization we saw was disguised as p-completion. It turns out that the category of p-complete spectra Sp_p is equivalent to the category of spectra which are $\mathbb{S}/p$-local, hence $\mathrm{X}_p \simeq L_{\mathbb{S}/p}(\mathrm{X})$.

We next discussed v_1-periodicity in the sphere, which lead to v_2-periodicity, and so on. This can also be phrased in terms of Bousfield localization. There are ring spectra $\mathrm{K}(n)$ known as *Morava K-theories*, whose homotopy groups take the form
$$\pi_*(\mathrm{K}(n)) \cong \mathbb{F}_p[v_n^{\pm 1}],$$
where $|v_n| = 2(p^n - 1)$. We also define $\mathrm{K}(0) = \mathbb{Q}$ and $\mathrm{K}(\infty) = \mathbb{F}_p$. The Morava K-theories act as field objects in Sp, and the v_n's act as "higher primes" or "intermediate characteristics":

- For any spectra X and Y, there is a Künneth isomorphism
$$\mathrm{K}(n)_*(\mathrm{X} \otimes \mathrm{Y}) \cong \mathrm{K}(n)_*(\mathrm{X}) \otimes_{\mathrm{K}(n)_*} \mathrm{K}(n)_*(\mathrm{Y}).$$
- If X is a $\mathrm{K}(n)$-module, then $\mathrm{X} \simeq \bigoplus_I (\mathrm{K}(n))[i]$. In other words, every $\mathrm{K}(n)$-module is free.
- If $n \neq m$, then $\mathrm{K}(n) \otimes \mathrm{K}(m) \simeq 0$.

In some way, the Bousfield localizations $L_{\mathrm{K}(n)}(-)$ give us access to v_n-periodicity (how good this is at seeing v_n-periodicity is precisely the question of the telescope conjecture [Rav84], which is now known to be false for $n \geqslant 2$ [BHLS23]).

Let $L_n(-) = L_{\mathrm{K}(0) \oplus \mathrm{K}(1) \oplus \cdots \oplus \mathrm{K}(n)}(-)$. By peeling off the $\mathrm{K}(n)$-summands one gets a sequence of natural transformations known as the *chromatic tower*
$$\cdots \to L_n \to L_{n-1} \to \cdots \to L_1 \to L_0 = L_{\mathbb{Q}}.$$
This tower has very nice properties. By the chromatic convergence theorem of Hopkins and Ravenel, if X is some finite p-complete spectrum, then the chromatic tower converges: there is an equivalence
$$\mathrm{X} = \lim_n(\cdots \to L_n\mathrm{X} \to L_{n-1}\mathrm{X} \to \cdots).$$
In particular, this applies for $\mathrm{X} = \mathbb{S}$! This tells us that we can reconstruct $\mathbb{S}$ from the chromatic localization $L_n\mathbb{S}$. One should think of this as an extension of the arithmetic fracture square (2.1) that encompasses higher periodic phenomena. We know that $L_0\mathbb{S} = \mathbb{S}_{\mathbb{Q}} \simeq \mathbb{Q}$, and so we move on to understanding $L_1\mathbb{S}$.

A useful fact about Bousfield localizations gives us some traction here. Let E and F be spectra, and let X be a spectrum whose F-localization is E-acyclic, so that $\mathrm{E}\otimes \mathrm{X}_{\mathrm{F}} \simeq 0$. Then there is a pullback square

$$\begin{tikzcd} \mathrm{X}_{\mathrm{E}\oplus\mathrm{F}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{X}_{\mathrm{E}} \arrow[d] \\ \mathrm{X}_{\mathrm{F}} \arrow[r] & (\mathrm{X}_{\mathrm{E}})_{\mathrm{F}} \end{tikzcd} \tag{3.1}$$

One can quite easily check that for any X, $\mathrm{K}(n) \otimes L_{n-1}\mathrm{X} \simeq 0$. Thus (3.1) gives us what is called the *chromatic fracture square*, taking the form

$$\begin{tikzcd} L_n\mathrm{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{X}_{\mathrm{K}(n)} \arrow[d] \\ L_{n-1}\mathrm{X} \arrow[r] & L_{n-1}(\mathrm{X}_{\mathrm{K}(n)}) \end{tikzcd}$$

Notice that the lefthand vertical map is the natural map coming from the chromatic tower. This fracture square is gluing data that allows us to inductively recover $L_n\mathrm{X}$. Namely, we must compute $\mathrm{X}_{\mathrm{K}(n)}$ and its L_{n-1}-localization, then glue this data together to get $L_n\mathrm{X}$.

Remark 3.3. Part of the motivation for why this process of reconstructing X from its chromatic localizations seems tractable is the *chromatic splitting conjecture*. Based on some actual computations, Hopkins conjectured the weak chromatic splitting conjecture (WCSC), which is that for $\mathrm{X} = \mathbb{S}$, the bottom map of the chromatic fracture square

$$L_{n-1}\mathbb{S} \to L_{n-1}(\mathbb{S}_{\mathrm{K}(n)})$$

splits. In other words, there is a section of this map realizing $L_{n-1}\mathbb{S}$ as a summand of $L_{n-1}(\mathbb{S}_{\mathrm{K}(n)})$. The fiber of this map was shown by Hovey to be equivalent to $\mathrm{map}(L_{n-1}\mathbb{S}, L_n\mathbb{S})$, so the weak CSC implies a splitting

$$L_{n-1}(\mathbb{S}_{\mathrm{K}(n)}) \simeq L_{n-1}\mathbb{S} \oplus \mathrm{map}(L_{n-1}\mathbb{S}, L_n\mathbb{S})[1].$$

This is known to hold for $n \leqslant 2$ and all primes p. There are many other forms of the CSC, which we may or may not discuss later.

Let's look back at understanding $L_1\mathbb{S}$. The relevant fracture square is

$$\begin{tikzcd} L_1\mathbb{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_{\mathrm{K}(1)} \arrow[d] \\ \mathbb{S}_{\mathbb{Q}} \simeq \mathbb{Q} \arrow[r] & (\mathbb{S}_{\mathrm{K}(1)})_{\mathbb{Q}} \end{tikzcd}$$

We understand the bottom left corner of this map quite well. Since $(\mathbb{S}_{\mathrm{K}(1)})_{\mathbb{Q}}$ is a module over $\mathbb{Q}$, we completely understand it as soon as we understand the torsion in $\mathbb{S}_{\mathrm{K}(1)}$. This leaves us with the task of understanding $\pi_*(\mathbb{S}_{\mathrm{K}(1)})$. And, in general, the task at hand is:

What are the homotopy groups of the $K(n)$-local sphere $\pi_*(\mathbb{S}_{\mathrm{K}(n)})$?

4. Descent

If we want to understand $\pi_*(\mathbb{S}_{\mathrm{K}(n)})$, then we need a computational tool.

Remark 4.1. This is where I am going to gloss over a lot of details. I am making choices here; they do not matter for our applications.

Let Γ_n denote the honda formal group law over $\mathbb{F}_{p^n}$, which is the unique formal group law over $\mathbb{F}_{p^n}$ such that its p-series si given by $[p]_{\Gamma_n}(x) = x^{p^n}$. The universal deformation of Γ_n is defined over the ring $(\mathrm{E}_n)_0 := \mathrm{W}(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!]$, where $\mathrm{W}(A)$ denotes the p-typical Witt vectors. By the Landweber Exact Functor Theorem, this data lifts to a complex-oriented spectrum E_n known as the *Lubin–Tate spectrum*, where $\pi_0(\mathrm{E}_n) \cong (\mathrm{E}_n)_0$ and $\pi_*(\mathrm{E}_n) \cong (\mathrm{E}_n)_0[u^{\pm 1}]$, $|u| = 2$. In other words,

$$\pi_*(\mathrm{E}_n) \cong \mathrm{W}(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}].$$

By the work of Goerss, Hopkins, and Miller, and later by Lurie, the Lubin–Tate spectrum is an $\mathbb{E}_\infty$-ring spectrum in $\mathrm{Sp}_{\mathrm{K}(n)}$. Moreover, they show that the automorphisms of Γ_n lift to spectrum level actions. Let $\mathbb{S}_n = \mathrm{Aut}(\Gamma_n, \mathbb{F}_p)$ be the automorphism group of Γ_n, considered over $\mathbb{F}_p$. Notice that there is a natural action of $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ on $\mathbb{S}_n$ by acting on the coefficients of any automorphism. We let $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ be the *Morava stabilizer group*. Equivalently, this is the automorphism group $\mathrm{Aut}(\Gamma_n, \mathbb{F}_{p^n})$. The work of the previous named authors shows that the action of $\mathbb{G}_n$ extends to an $\mathbb{E}_\infty$-action on the Lubin–Tate spectrum E_n, and that the $\mathrm{K}(n)$-local unit map

$$\mathbb{S}_{\mathrm{K}(n)} \to \mathrm{E}_n$$

is $\mathbb{G}_n$-equivariant, where we endow $\mathbb{S}_{\mathrm{K}(n)}$ with the trivial action.

Devinatz and Hopkins showed that we are in probably the best place we have ever been, in terms of computing $\pi_*(\mathbb{S}_{\mathrm{K}(n)})$ [DH04]. The unit map $\mathbb{S}_{\mathrm{K}(n)} \to \mathrm{E}_n$ is actually $\mathbb{G}_n$-Galois (technically pro-Galois since the Morava stablizer group is profinite). This implies that there are equivalences $\mathbb{S}_{\mathrm{K}(n)} \simeq \mathrm{E}_n^{h\mathbb{G}_n}$ and $\mathrm{E}_n \widehat{\otimes} \mathrm{E}_n \simeq \mathrm{map}_{\mathrm{cts}}(\mathbb{G}_n, \pi_*(\mathrm{E}_n))$, where $\widehat{\otimes}$ denotes the $\mathrm{K}(n)$-local tensor product. It is really the first point that we will focus on here, as it implies that there is a convergence *descent spectral sequence* taking the form

$$\mathrm{E}_2^{s,f} = H^f_{\mathrm{cts}}(\mathbb{G}_n; \pi_{s+f}(\mathrm{E}_n)) \implies \pi_s(\mathbb{S}_{\mathrm{K}(n)}).$$

Here the cohomology is continuous, as it is taken with respect to the topology on $\mathbb{G}_n$. It's worth mentioning that the DSS is also (clearly) a homotopy fixed point spectral sequence, and that it is also (not so clearly) the $\mathrm{K}(n)$-local E_n-based Adams spectral sequence.

This spectral sequence is famously difficult to compute with. Computing the E_2-page is a difficult task itself, and determining differentials a whole other can of worms. However, instead of taking fixed points with respect to the entire Morava stabilizer group, we could try instead to take fixed points with respect to smaller, finite subgroups $G \leqslant \mathbb{G}_n$. This gives rise to a *homotopy fixed point spectral sequence*

$$\mathrm{E}_2^{s,f} = \mathrm{H}^f(G; \pi_{s+f}(\mathrm{E}_n)) \implies \pi_s(\mathrm{E}_n^{hG}).$$

We will refer to the spectra $\mathrm{EO}_n(G) := \mathrm{E}_n^{hG}$ as *higher real K-theories*, whose naming will be justified in the next section.

The whole point of these higher real K-theories is that they provide a factorization of the unit map for the Lubin–Tate spectrum

$$\begin{tikzcd} \mathbb{S}_{\mathrm{K}(n)} \arrow[rr] \arrow[dr] & & \mathrm{E}_n \\ & \mathrm{EO}_n(G) \arrow[ur] & \end{tikzcd}$$

where the map $\mathrm{EO}_n(G) \to \mathrm{E}_n$ is the inclusion of fixed points. In this sense, $\mathrm{EO}_n(G)$ is literally "closer to the sphere" than E_n is. Moreover, the map $\mathbb{S}_{\mathrm{K}(n)} \to \mathrm{EO}_n(G)$ is still Galois, so we have a chain of equivalences

$$\mathbb{S}_{\mathrm{K}(n)} \simeq \mathrm{E}_n^{h\mathbb{G}_n} \simeq (\mathrm{E}_n^{hG})^{h(\mathbb{G}_n/G)} \simeq \mathrm{EO}_n(G)^{h(\mathbb{G}_n/G)},$$

Thus we may still completely descend to $\mathbb{S}_{\mathrm{K}(n)}$ from $\mathrm{EO}_n(G)$.

5. The K(1)-local sphere and the L_1-local sphere

Recall that to understand $L_1\mathbb{S}$, we needed to understand $\mathbb{S}_{\mathrm{K}(1)}$ and its rationalization. We now have some tools to compute with, and let's see where they get us!

Remark 5.1. This is a bit of a digression from my actual talk, but it is too worthwhile to not contribute. Also, the DSS is actually not too hard to compute in these cases, but it is more fun to do it in a piecemeal fashion.

5.1. p odd.

Suppose that the implicit prime p is odd. Then an explicit model for the Lubin–Tate spectrum is $\mathrm{E}_1 = \mathrm{KU}$, the ($p$-complete) complex K-theory spectrum. Notice that since $\pi_*(\mathrm{KU}) \cong \mathbb{Z}_p[u^{\pm 1}]$ and all higher homotopy groups of $\mathbb{S}$ are torsion, the image of the unit map $\mathbb{S}_{\mathrm{K}(1)} \to \mathrm{KU}$ is only nonzero in degree 0, and it is indeed an isomorphism there.

There is an isomorphism $\mathbb{G}_1 \cong \mathbb{Z}_p^\times$, the units in the p-adic integers (in fact, $\mathbb{G}_1 \cong \mathbb{S}_1$!). This group decomposes as $C_{p-1} \times \mathbb{Z}_p$, where the cyclic factor corresponds to those units congruent to 1 mod p. Since $\mathbb{Z}_p$ admits no finite subgroups, this C_{p-1} is a maximal finite subgroup of $\mathbb{G}_1$, giving us a decomposition

$$\mathbb{S}_{\mathrm{K}(1)} \simeq (\mathrm{KU}^{hC_{p-1}})^{h\mathbb{Z}_p}.$$

For any $\alpha \in \mathbb{Z}_p^\times$, the action of α on the polynomial generator of $\pi_*(\mathrm{KU})$ is given by multiplication, so that $\alpha \cdot u = \alpha u$.

Let's compute the first HFPSS

$$\mathrm{E}_2^{s,f} = H^f(C_{p-1}, \pi_{s+f}(\mathrm{KU})) \implies \pi_s(\mathrm{KU}^{hC_{p-1}}).$$

Notice that $p-1$ is invertible in $\mathbb{Z}_p^\times$, so the cohomology vanishes in positive filtration. The only nontrivial contributions to the 0-line come from the instances where C_{p-1} acts trivially, hence

$$H^*(C_{p-1}, \pi_*(\mathrm{KU})) \cong \mathbb{Z}_p[u^{\pm(p-1)}].$$

Since the spectral sequence is concentrated in filtration 0, it collapses for degree reasons. Let $v_1 := u^{p-1}$. Then there is an isomorphism

$$\pi_*(\mathrm{KU}^{hC_{p-1}}) \cong \mathbb{Z}_p[v_1].$$

Note that often this spectrum is called the *Adams summand* and denoted L. It is also worth mentioning that the image of the unit map $\mathbb{S} \to \mathrm{L}$ is just as small as with KU, but that also we did nothing in this spectral sequence since it collapses at E_2, so maybe we shouldn't expect that. Another justification for this is that L is literally a summand of KU: there is a splitting

$$\mathrm{KU} \simeq \bigoplus_{i=0}^{p-2} \mathrm{L}[2i].$$

Now, our goal is to get to $\mathbb{S}_{\mathrm{K}(1)}$. A particularly nice way to get there which involves running the entire $\mathbb{Z}_p^\times$-homotopy fixed point spectral sequence is by a long exact sequence in homotopy groups. If we let $\psi \in \mathbb{Z}_p^\times$ be a topological generator, then one can compute the homotopy fixed points $(\mathrm{KU})^{h\mathbb{Z}_p^\times} \simeq \mathbb{S}_{\mathrm{K}(1)}$ as the equalizer of $\psi - 1$. In other words, there is a cofiber sequence

$$\mathbb{S}_{\mathrm{K}(1)} \to \mathrm{KU} \xrightarrow{\psi-1} \mathrm{KU}$$

However, by breaking up the fixed points as we have, we could also choose a generator $\psi \in \mathbb{Z}_p \cong \mathbb{Z}_p^\times / C_{p-1}$, resulting in a cofiber sequence

$$\mathbb{S}_{\mathrm{K}(1)} \to \mathrm{L} \xrightarrow{\psi-1} \mathrm{L}.$$

This is quite nice, actually. It turns out that no matter the choice of ψ, the result of this map on the generator v_1 is

$$\psi(v_1) = (1+p)v_1^{p-1}.$$

This completely determines the cofiber sequence. Therefore,

$$\pi_s(\mathbb{S}_{\mathrm{K}(1)}) \cong \begin{cases} \mathbb{Z}_p & s = -1, 0 \\ \mathbb{Z}/p^{\nu_p(k)+1} & s = 2k(p-1)-1 \\ 0 & \text{else.} \end{cases}$$

This implies that when we rationalize, we get

$$\pi_s((\mathbb{S}_{\mathrm{K}(1)})_{\mathbb{Q}}) \cong \begin{cases} \mathbb{Q}_p & s = -1, 0 \\ 0 & \text{else.} \end{cases}$$

Now, we know all the corners of the chromatic fracture square and can compute $\pi_*(L_1\mathbb{S})$! There is a long exact sequence in homotopy groups

$$\cdots \to \pi_s(L_1\mathbb{S}) \to \pi_s(\mathbb{S}_{\mathbb{Q}}) \oplus \pi_s(\mathbb{S}_{\mathrm{K}(1)}) \to \pi_s((\mathbb{S}_{\mathrm{K}(1)})_{\mathbb{Q}}) \to \cdots$$

Using what we have already computed, it is straightforward to piece together $\pi_*(L_1\mathbb{S})$ using just this LES. The result is

$$\pi_s(L_1\mathbb{S}) \cong \begin{cases} \mathbb{Q}_p/\mathbb{Z}_p & s = -2 \\ \mathbb{Z}_p & s = 0 \\ \mathbb{Z}/p^{\nu_p(k)+1} & s = 2k(p-1)-1 \\ 0 & \text{else.} \end{cases}$$

5.2. $p = 2$.

Things are slightly different when $p = 2$. We have that $\mathrm{E}_1 = \mathrm{KU}$, the (2-complete) complex K-theory spectrum, and $\mathbb{G}_1 = \mathbb{Z}_2^\times$, the 2-adic units. However, the product decomosotion is slightly different. There is an isomorphism $\mathbb{Z}_2^\times = C_2 \times \mathbb{Z}_2$, where C_2 correseponds to those units congruent to 1 mod 4. Again, C_2 is a maximal finite subgroup, so that

$$\mathbb{S}_{\mathrm{K}(1)} \simeq (\mathrm{KU}^{hC_2})^{h\mathbb{Z}_2}.$$

The action of any $\alpha \in \mathbb{Z}_2^\times$ is the same as in the odd primary case.

Let's compute the first HFPSS

$$\mathrm{E}_2^{s,f} = \mathrm{H}^f(C_2, \pi_{s+f}(\mathrm{KU})) \implies \pi_s(\mathrm{KU}^{hC_2}).$$

We do not get such a simple reduction of the E_2-page as we did in the odd primary case. If we let $\sigma \in C_2$ be a generator, then $\sigma \cdot u = -u$. This describes the entire action of C_2 on $\pi_*(\mathrm{KU})$, and we can compute cohomology by explicit resolutions:

$$H^*(C_2; \pi_s(\mathrm{KU})) = \begin{cases} \mathbb{Z}[x]/(2x),\ |x| = 2 & s \equiv 0\,(4) \\ \bigoplus_{i \geqslant 0} \mathbb{Z}/2[2i+1] & s \equiv 2\,(4). \end{cases}$$

This weird congruence mod 4 just remembers that the polynomial generator in $\pi_*(\mathrm{KU})$ is in degree 2.

These cohomology groups assemble to describe the E_2-page as an algebra:

$$\mathrm{E}_2^{*,*} = \mathbb{Z}[u^{\pm 2}, a]/(2a),$$

where $|u^2| = (4, 0)$ and $|a| = (1, 1)$. There is no room for any d_2-differentials in the spectral sequence. To determine the d_3-differentials, we can finally reveal the big secret of why $\mathrm{EO}_n(G)$ are called higher real K-theories: basically by the work of Atiyah, there is an equivalence

$$\mathrm{EO}_1(C_2) \simeq \mathrm{E}_1^{hC_2} \simeq \mathrm{KO},$$

the real topological K-theory spectrum! Supposing we already knew that $\eta^3 = 0$ in KO and that the class a must realize to η (which one can show by using the Adams–Novikov), we can determine that there must be a differential

$$d_3(u^2) = a^3.$$

The Leibniz rule extends this to the entire E_3-page, and for degree reasons there are no further differentials. This gives us an isomorphism

$$\pi_*(\mathrm{KO}) \cong \frac{\mathbb{Z}[\eta, x, v_1^4]}{(2\eta, \eta^3, \eta x, x^2 - 4v_1^4)},$$

where $|\eta| = 1$ and is the name we give to a, $|x| = 4$ and is the name we give to $2u^2$, and $|v_1^4| = 8$ and is the name we give to u^4. Notice that the Hurewicz image of KO is larger than that of KU (we see $v_1^{4n}\eta$, and $v_1^{4n}\eta^2$), and that there is no 2-primary decomposition of KU into sums of shifts of KO.

We can get to $\mathbb{S}_{\mathrm{K}(1)}$ by the same argument as in the odd primary case. There is a cofiber sequence of spectra

$$\mathbb{S}_{\mathrm{K}(1)} \to \mathrm{KO} \xrightarrow{\psi - 1} \mathrm{KO}$$

where $\psi \in \mathbb{Z}_2$ is a topological generator. Since we are working in the 2-complete setting, there is a lot of freedom here for choices of generator and they don't end up mattering.

Common ones are 3 or 5, and we'll choose $\psi = 3$. The effect of ψ on the coefficients of KO can be computed easily:

$$\psi(\eta) = 0, \quad \psi(x) = 3^2 x, \quad \psi(v_1^4) = 3^4 v_1^4.$$

Running the long exact sequence in homotopy gives our result:

$$\pi_s(\mathbb{S}_{\mathrm{K}(1)}) \cong \begin{cases} \mathbb{Z}_2 & s = -1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}/2 & s = 0 \\ \mathbb{Z}/2 & s \equiv 0, 2\,(8),\ s > 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & s \equiv 1\,(8) \\ \mathbb{Z}/8 & s \equiv 3\,(8) \\ \mathbb{Z}/2^{\nu_v(k)+4} & s = -1 + 8k,\ k \neq 0 \\ 0 & \text{else.} \end{cases}$$

Figure 4.14 of [BB20] is a great depiction of this jazz.

This implies that when we rationalize, we get

$$\pi_s((\mathbb{S}_{\mathrm{K}(1)})_{\mathbb{Q}}) \cong \begin{cases} \mathbb{Q}_2 & s = -1, 0 \\ 0 & \text{else.} \end{cases}$$

Using the long exact sequence associated to the chromatic fracture square, we can similarly compute $L_1\mathbb{S}$:

$$\pi_s(L_1\mathbb{S}) \cong \begin{cases} \mathbb{Q}_2/\mathbb{Z}_2 & s = -2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}/2 & s = 0 \\ \mathbb{Z}/2 & s \equiv 0, 2\,(8),\ s > 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & s \equiv 1\,(8) \\ \mathbb{Z}/8 & s \equiv 3\,(8) \\ \mathbb{Z}/2^{\nu_v(k)+4} & s = -1 + 8k,\ k \neq 0 \\ 0 & \text{else.} \end{cases}$$

Remark 5.2. Said differently, regardless of the prime, we have a decomposition of $\pi_*(L_1\mathbb{S})$ into a $\mathbb{Q}_p/\mathbb{Z}_p$ in degree -2, a $\mathbb{Z}_p$ in degree 0, and all of the torsion components of $\pi_*(\mathbb{S}_{\mathrm{K}(1)})$ in precisely the same degrees.

Remark 5.3. The homotopy groups of $\pi_*(\mathbb{S}_{\mathrm{K}(1)})$ account for more than just the v_1-periodic elements of the sphere. An easy way to see this is that $\mathbb{S}_{\mathrm{K}(1)}$ has nontrivial homotopy in degree -1, whereas the sphere is connective. However, one can determine that the localization map

$$\mathbb{S} \to \mathbb{S}_{\mathrm{K}(1)}$$

is an isomorphism on v_1-periodic components in positive degrees.

6. Higher heights

So we have computed $L_1\mathbb{S}$. What about $L_n\mathbb{S}$ for arbitrary n?

Let's consider just $L_2\mathbb{S}$. The chromatic fracture square takes the form

$$\begin{tikzcd} L_2\mathbb{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_{\mathrm{K}(2)} \arrow[d] \\ L_1\mathbb{S} \arrow[r] & L_1(\mathbb{S}_{\mathrm{K}(2)}) \end{tikzcd}$$

A natural starting place is to compute $\mathbb{S}_{\mathrm{K}(2)}$. The DSS takes the form

$$\mathrm{E}_2^{s,f} = H^f_{\mathrm{cts}}(\mathbb{G}_2, \pi_{s+f}(\mathrm{E}_2)) \implies \pi_s(\mathbb{S}_{\mathrm{K}(2)}).$$

The profinite group $\mathbb{G}_2$ is far more complicated than $\mathbb{G}_1$, and so we should really switch to the higher real K-theory philosophy. That is, we should approximate $\mathbb{S}_{\mathrm{K}(2)}$ by computing various HFPSS's

$$\mathrm{E}_2^{s,f} = H^f(G, \pi_{s+f}(\mathrm{E}_2)) \implies \pi_s(\mathrm{EO}_2(G)),$$

where $G \subseteq \mathbb{G}_2$ is a finite subgroup.

I'm not going to be encyclopedic here, but there is so much work on this question. There are many subgroups of $\mathbb{G}_2$ worth considering, and many people have done some awesome computations. For a nice list, one should check the bottom of page 4 of [DHLLSWX25]. Some key spectra that show up in height 2 computations are the various spectra of topological modular forms. Related computations are the Hurewicz image of various height 2 higher real K-theories (which see more v_2-periodicity than E_2 does), and various resolutions of the K(2)-local sphere (which are higher height analogues of the cofiber sequence $\mathbb{S}_{\mathrm{K}(1)} \to \mathrm{KU} \to \mathrm{KU}$).

At heights higher than 2, there has primarily been work at odd primes. Various sources scattered through the literature have done some computations, but I will leave those for the interested reader to find (and as a task, find the very very few computations valid at all chromatic heights).

Part 2. Redshift and Blueshift

7. Introduction

The chromatic tower gives us a way to reconstruct $\pi_*(\mathbb{S})$ from $\pi_*(L_n\mathbb{S})$. One part of motivation for this tower is the presence of higher periodic phenomena; the L_n-local and $\mathrm{K}(n)$-local spheres are supposed to be related to the v_n-periodic component of $\pi_*(\mathbb{S})$. However, this doesn't quite do the job. Let's set up some background.

An important structural result in stable homotopy theory is the Thick Subcategory Theorem of Devinatz, Hopkins, and Smith [DHS88; HS98]. This theorem states that $\mathrm{Spc}(\mathrm{Sp}_p^\omega)$, the Balmer spectrum of the category of p-complete finite spectra, has exactly one point $\mathcal{C}_n$ for each $n \geqslant 0$, and that the topology is the linear order topology. In other words, $\{\mathcal{C}_0\}$ is open, $\{\mathcal{C}_0, \mathcal{C}_1\}$ is open, $\{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2\}$ is open, and so on. Building on this result is the Periodicity Theorem. If $\mathrm{X} \in \mathcal{C}_n \backslash \mathcal{C}_{n-1}$ is some finite spectrum, then there is a non-nilpotent v_n-self-map

$$v_n : \mathrm{X}[d] \to \mathrm{X}.$$

This is a map which induces an isomorphism in $\mathrm{K}(n)$-homology. We define the telescope $\mathrm{T}(n)$ as

$$\mathrm{T}(n) = \mathrm{colim}_n(\mathrm{X} \xrightarrow{v_n} \mathrm{X}[-d] \xrightarrow{v_n} \cdots).$$

As it turns out, the Bousfield class of $\mathrm{T}(n)$ is independent on the choice of finite spectrum X and v_n-self-map, so that $\mathrm{T}(n)$-localization is well-defined.

Remark 7.1. By construction, $\mathrm{T}(n)$-localization brings us closer to v_n-periodicity. We also may make the same argument for $\mathrm{K}(n)$-localization. Indeed, there is a natural map

$$\mathbb{S}_{\mathrm{K}(n)} \to \mathbb{S}_{\mathrm{T}(n)}, \tag{7.1}$$

Note that this map exists for any spectrum, not just $\mathbb{S}$. The question of whether or not this map is an equivalence is precisely the statement of the telescope conjecture. The case $n = 0$ is trivial as $\mathrm{K}(0) = \mathrm{T}(0) = \mathbb{Q}$. The case $n = 1$ was proven by Mahowald for $p = 2$ [Mah81] and by Miller for p odd [Mil81]. The case $n \geqslant 2$ was recently disproven [BHLS23].

We note that it is sad that the telescope conjecture is not true. We have a lot of algebraic machinery to compute $\pi_*(\mathbb{S}_{\mathrm{K}(n)})$, and much less to compute $\pi_*(\mathbb{S}_{\mathrm{T}(n)})$. However, if we are focused on periodicity, it seems that the telescopic localization is more directly tied to v_n-periodicity. We also note that while the full extent of the failure of the telescope conjecture is not understood, it is known that the map (7.1) is neither injective nor surjective. For example, the image of $\zeta \in \pi_{-1}(\mathbb{S}_{\mathrm{K}(n)})$ is trivial, and there are classes $\tau_k \in \pi_0(\mathbb{S}_{\mathrm{T}(n)})$ which are not in the image.

One can use $\mathrm{T}(n)$-homology as a type of detector of the chromatic behavior of a ring spectrum. A theorem of Jeremy Hahn says that if $\mathrm{R} \in \mathrm{CAlg}(\mathrm{Sp})$ such that $\mathrm{R} \otimes \mathrm{T}(n) \simeq 0$, then $\mathrm{R} \otimes \mathrm{T}(n+1) \simeq 0$. Then, we define the *height* of R to be

$$\mathrm{height}(R) := \max\{n \geqslant -1 : R \otimes \mathrm{T}(n) \neq 0\}.$$

For example,

- $\mathrm{height}(\mathbb{F}_p) = -1$, as $\mathrm{T}(0) \otimes \mathbb{F}_p = \mathbb{Q} \otimes \mathbb{F}_p \simeq 0$.
- $\mathrm{height}(\mathbb{Z}_p) = 0$, as every class in $\pi_*(\mathbb{Z}_p)$ is concentrated in degree 0, forcing every class to be v_1-torsion.

- height(ku) = 1, as, well, if you reduce it mod-p you get a summand of the connective cover of K(1) (this is imprecise but whatever).

Here is a question one may ask:

Suppose that height(R) $= n$. *Is there a functor* $F : \mathrm{CAlg}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp})$ *such that* height$(F(\mathrm{R})) \neq n$?

8. Redshift

A useful construction we can perform to a ring R is to take its algebraic K-theory K(R). If R were $\mathbb{E}_\infty$, then so too is K(R). An interesting finding in computations with algebraic K-theory is its strange relationship with height. For example,

- Quillen's computation of the algebraic K-theory of finite fields shows that $\mathrm{K}(\mathbb{F}_p)_p \simeq \mathbb{Z}_p$, hence height$(\mathrm{K}(\mathbb{F}_p)) = 0$.
- Work of Steve Mitchell shows that $\mathrm{K}(\mathbb{Z}_p)_{\mathrm{T}(1)} \neq 0$ while $\mathrm{K}(\mathbb{Z}_p)_{\mathrm{T}(n)} = 0$ for $n \geqslant 2$, hence height$(\mathrm{K}(\mathbb{Z}_p)) = 1$.
- Work of Ausoni and Rognes shows that $\mathrm{K}(\mathrm{ku})_{\mathrm{T}(2)} \neq 0$ while $\mathrm{K}(\mathrm{ku})_{\mathrm{T}(n)} = 0$ for $n \geqslant 3$, hence height(K(ku)) = 2.

These observations seem to suggest that algebraic K-theory raises chromatic height by 1, i.e. it exhibits *redshift* by 1. This was originally conjectured by Rognes, based on his work with Ausoni. Recent work of Yuan has proven this for $\mathrm{R} = \mathrm{E}_n$, and follow-up work of Burklund, Schlank, and Yuan shows this to be true for all $\mathbb{E}_\infty$-rings [BSY22].

It is still nice to have a handle on how to actually see the v_{n+1}-periodicity of the K-theory of a height n spectrum R. As an example, I'll briefly go over the work of Angelini-Knoll, Ausoni, and Rognes [AAR23]. Their work is a study of the homotopy of K(ko). To make matters a little easier, since ko detects $2, \eta$, and v_1^4, they compute K(ko) modulo $2, \eta$, and v_1^4.

Remark 8.1. To elaborate a little bit on this, because it's really cool, they actually compute the A(1)-homology of K(ko). Here A(1) is an 8-cell complex constructed in the following way. Consider the mod-2 Moore spectrum

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \to \mathbb{S}/2.$$

One can show (essentially because of torsion reasons) that there is a lift of η to $\mathbb{S}/2$. Let Y be the cofiber

$$\mathbb{S}/2[1] \xrightarrow{\eta} \mathbb{S}/2 \to \mathrm{Y}.$$

Davis and Mahowald showed that Y has a v_1-self-map of periodicity 1 (which is better than the v_1^4-self-map on $\mathbb{S}/2$) [DM81]. Let A(1) be the cofiber

$$\mathrm{Y}[2] \xrightarrow{v_1} \mathrm{Y} \to \mathrm{A}(1).$$

By tracing through the long exact sequences in cohomology, one can see that $\mathrm{H}^*(\mathrm{A}(1))$ is of rank 8 over $\mathbb{F}_2$ and is isomorphic to $\mathcal{A}(1)$ as a module over $\mathcal{A}(1)$. There are, however, many potential nontrivial Sq^2-actions. We also note that the minimal v_2-self-map on A(1) is of periodicitiy 32 [BEM17].

Briefly, the method of computation for K(ko) is by *trace methods*. The *Dennis trace* is a map of ring spectra

$$\mathrm{K}(\mathrm{ko}) \to \mathrm{THH}(\mathrm{ko}),$$

which approximates K-theory by topological Hochschild homology. One can do more, i.e. get closer to K-theory, by instead looking at the *cyclotomic trace*

$$\mathrm{K}(\mathrm{ko}) \to \mathrm{TC}(\mathrm{ko}),$$

which factors the Dennis trace through topological cyclic homology. Justfication for why this is a good method to approximate K-theory is the following (which can be found somewhere in DGM): if $\mathrm{R} \in \mathrm{CAlg}(\mathrm{Sp})$ and there is a surjection $\mathbb{Z}_p \to \pi_0(\mathrm{R})$, then there is a cofiber sequence

$$\mathrm{K}(\mathrm{R}) \to \mathrm{TC}(\mathrm{R}) \to \mathbb{Z}_p[-1].$$

In other words, apart from a small $\mathbb{Z}_p$-summand, these two invariants are practically the same thing.

The computation of TC(ko) is done via the motivic spectral sequence

$$\mathrm{gr}^*_{\mathrm{mot}}(\mathrm{TC}(\mathrm{ko})) \implies \mathrm{TC}(\mathrm{ko})$$

coming from the even filtration [HRW24]. I won't get into the details of their methods, but it is worth remarking that the previous computation of K(ku) due to Ausoni and Rognes comes really in handy here.

There are a lot of nice results coming from these computations. First, the associated graded for the motivic filtration on TC(ko) is often referred to as *syntomic cohomology*. The authors show that the mod-$(2, \eta, v_1)$ syntomic cohomology of ko is a free $\mathbb{F}_2[v_2]$-module on 14 generators. They also leverage the DGM cofiber sequence to show that there is an exact sequence of $\mathbb{Z}/4[v_2^{32}]$-modules

$$0 \to \mathbb{F}_2[1] \oplus \mathbb{F}_2[3] \to \mathrm{A}(1)_*(\mathrm{K}(\mathrm{ko})) \to \mathrm{A}(1)_*(\mathrm{TC}(\mathrm{ko})) \to \mathbb{F}_2\{x, y\} \to 0.$$

This gives very clearly the relationship between TC and K-theory.

Another interesting result that they give is related to the Burklund–Schlank–Yuan proof of redshift. In their proof, they show that if $\mathrm{height}(\mathrm{R}) = n$, then there is a map of ring spectra $\mathrm{R} \to \mathrm{E}_n$. In particular, since $\mathrm{height}(\mathrm{K}(\mathrm{ko})) = 2$ (as is evidenced by the nontrivial v_2-actions above!), there is a map $\mathrm{K}(\mathrm{ko}) \to \mathrm{E}_2$. However, the algebraic K-theory spectrum is connective while E_2 is periodic. One might ask if there is a map of rings from K(ko) to some connective height 2 ring, related to E_2, such as tmf. Angelini-Knoll–Ausoni–Rognes show that there is no map $\mathrm{K}(\mathrm{ko}) \to \mathrm{tmf}$ which respects the $\mathbb{E}_n$-structure for any $n \geqslant 0$. Weird!

The last thing worth mentioning is that the spectra considered here do not break the telescope conjecture: there are equivalences

$$\mathrm{K}(\mathrm{ko})_{\mathrm{K}(2)} \xrightarrow{\simeq} \mathrm{K}(\mathrm{ko})_{\mathrm{T}(2)}, \quad \mathrm{TC}(\mathrm{ko})_{\mathrm{K}(2)} \xrightarrow{\simeq} \mathrm{TC}(\mathrm{ko})_{\mathrm{T}(2)}.$$

As these are height 2 rings, this is not immediately obvious. The disproof of the telescope conjecture is obtained by showing that

$$\mathrm{K}(\mathrm{BP}\langle n\rangle^{hp\mathbb{Z}})_{\mathrm{K}(n+1)} \to \mathrm{K}(\mathrm{BP}\langle n\rangle^{hp\mathbb{Z}})_{\mathrm{T}(n+1)}$$

is not an equivalence for $n \geqslant 1$. Since $\mathrm{BP}\langle 1\rangle \simeq \mathrm{ku}$ and $\tau_{\geqslant 0}\mathrm{ku}^{hC_2} \simeq \mathrm{ko}$, one might expect K(ko) to not satisfy the telescope conjecture. Surprise!

9. Blueshift

Let X be a spectrum with G-action. We have seen that the homotopy fixed points X^{hG} are often interesting. Recall that we may consider X as a functor $\mathrm{X} : BG \to \mathrm{Sp}$, and this allows us to define the homotopy fixed points as

$$\mathrm{X}^{hG} = \mathrm{map}(EG_+, \mathrm{X}) = \lim(\mathrm{X} : BG \to \mathrm{Sp}).$$

A dual construction to the homotopy fixed points are the *homotopy orbits*, which can be defined as

$$\mathrm{X}_{hG} = EG_+ \otimes \mathrm{X} = \mathrm{colim}(\mathrm{X} : BG \to \mathrm{Sp}).$$

There is a canonical map from the colimit to the limit of X, and the cofiber of this map are the *Tate fixed points*. So, we have a cofiber sequence

$$\mathrm{X}_{hG} \to \mathrm{X}^{hG} \to \mathrm{X}^{tG}.$$

Sometimes, the Tate fixed points are not very interesting.

For example, let's take the conjugation action of C_2 on KU. There is a spectral sequence (the Tate spectral sequence) taking the form

$$\mathrm{E}_2 = \widehat{\mathrm{H}}^*(C_2; \pi_*(\mathrm{KU})) \implies \pi_*(\mathrm{KU}^{tC_2}),$$

where the E_2-page is just Tate cohomology. It is not hard to show that one may obtain the Tate E_2-page by inverting a positive filtration degree element of the HFPSS E_2-page which acts by isomorphisms. So, remembering that spectral sequence from above, the E_2-page of the TSS takes the form

$$\mathrm{E}_2 = \mathbb{F}_2[u^{\pm 2}, a].$$

The canonical map $\mathrm{KU}^{hC_2} \to \mathrm{KU}^{tC_2}$ pushes the differentials in the HFPSS to the TSS. In fact, this d_3-differential extends to the entire spectral sequence and annihilates everything in sight! In other words, we get that $\mathrm{KU}^{tC_2} \simeq *$.

However, in the presence of a *trivial* group action, strange things can happen. Here are some motivating examples. In all of these, let Σ_2 act trivially.

- Davis and Mahowald showed that
$$\mathrm{ko}^{t\Sigma_2} \simeq \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}[4j]$$
- Greenlees and May showed that
$$\mathrm{KO}^{t\Sigma_2} \simeq \bigoplus_{j \in \mathbb{Z}} \mathbb{Q}[4j]$$
- Greenlees and Sadofsky showed that
$$\mathrm{K}(n)^{tG} \simeq *$$
for any finite group G and any action!
- Bailey and Ricka showed that
$$\mathrm{tmf}^{t\Sigma_2} \simeq \bigoplus_{j \in \mathbb{Z}} \mathrm{ko}[4j].$$

Other than the third point, we start to see a pattern: if $\text{height}(\mathrm{R}) = n$, then $\text{height}(\mathrm{R}^{t\Sigma}) \leqslant n-1$. This phenomenon is called *blueshift*. In someinstances we have seen chromatic height lower by 1, and in some instances we have seen chromatic height lower all the way to 0. For instance, I am pretty sure that $\mathrm{E}_n^{tG} \simeq 0$ for any finite $G \subseteq \mathbb{G}_n$. Hoewver, one thing that is consistent is that if $\mathrm{R} \in \mathrm{Sp}_{\mathrm{K}(n)}$, then $(\mathrm{R}^{tG})_{\mathrm{K}(n)} \simeq *$. This is often referred to as $\mathrm{K}(n)$-local Tate vanishing.

I'll just end by saying that blueshift is really hard to understand. Yuan's proof of redshift for E_n does say something substantial, at least. It is more of a "redshift is un-blueshift" type of statement: if $\mathrm{R} \in \mathrm{CAlg}(\mathrm{Sp})$ such that $\mathrm{R}_{\mathrm{T}(n)} \not\simeq *$, then $\mathrm{K}(\mathrm{R}^{tC_p})_{\mathrm{T}(n)} \not\simeq *$. Of similar interest is that Tate blueshift only seems to be a statement about finite group actions, as Yuan shows that the same hypothesis on R shows that $\mathrm{R}^{tS^1}_{\mathrm{T}(n)} \not\simeq *$.

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