GALOIS DESCENT AND THE PICARD GROUP OF K-THEORY

JACKSON MORRIS

Abstract. In this talk, we introduce the Picard group from a homotopical point of view. We first define the Picard group of a symmetric monoidal category, then generalize to the Picard spectrum of a commutative ring spectrum. Our primary tool of computation is Galois descent. As applications, we compute the Picard group of classical rings and of topological K-theory spectra.

Contents

1. Introduction

1.1. **Motivation from algebraic geometry.** Let's start with some classical algebraic geometry. Let X be some scheme. The *Picard group* of X, denoted here $\mathrm{Pic}(X)$, is the group of isomorphism classes of invertible sheaves, or line bundles, on X under the tensor product. For $X = \mathrm{Spec}(A)$ affine, we often denote the Picard group as $\mathrm{Pic}(A)$. This is a very useful invariant, which we highlight by some examples below.

Example 1.1. If $X = \mathrm{Spec}(A)$ where A is a Dedekind domain, then $\mathrm{Pic}(A)$ is isomorphic to the ideal class group of A. Moreover, there is an isomorphism:

$$\mathrm{K}_0(A) \cong \mathbb{Z} \oplus \mathrm{Pic}(A).$$

Example 1.2. If X is a Noetherian, integral, separated locally factorial scheme, then $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$, the class group of divisors on X.

Example 1.3. If A is the ring of integers of an algebric number field, then A is a UFD if and only if $\mathrm{Pic}(A) = 0$. In this case, the Picard group is the same as the ideal class group of A.

I could keep going, but this quick list of examples surely shows one thing: the Picard group is important and worth studying. One way that we can calculate the Picard group is by identifying it in terms of sheaf cohomology group:

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*).$$

This identification gives a bucketload of tricks which we can use. However, we will take a different perspective on the matter.

The *derived Picard group* of a ring A, denoted $\mathrm{DPic}(A)$, is an derived invariant related to the Picard group and defined analogously. The key property, for us, is the following.

Proposition 1.4. [Yek99, Proposition 3.5] *Let A be a commutative ring. Then*

$$\mathrm{DPic}(A) \cong \mathbb{Z}^m \times \mathrm{Pic}(A),$$

where m is the number of connected components of $\mathrm{Spec}(A)$.

Thus, at least in the affine case, we can recover the Picard group from the derived Picard group if we understand the geometry of $\mathrm{Spec}(A)$. The derived Picard group is, from the point of view of homotopy theory, much more computable. We give a brief overview of this below.

1.2. **Spectra.** In stable homotopy theory, we work most often with the category of spectra Sp. The objects of this category represent cohomology theories. This category behaves much like a larger version of the derived category $\mathcal{D}(\mathbb{Z})$:

- This is a symmetric monoidal category, and the monoidal unit is denoted by $\mathbb{S}$;
- This is a triangulated category, with triangles (called here *cofiber sequences*) denoted
$$\mathrm{A} \to \mathrm{B} \to \mathrm{C} \to \Sigma\mathrm{A};$$
- The objects of Sp are a lot like chain complexes, where the process of taking cohomology is replaced with taking homotopy groups. More precisely, there is a t-structure on Sp given by
$$\mathrm{Sp}_{\geq 0} = \{\mathrm{X} \in \mathrm{Sp} : \pi_i(X) = 0 \text{ for all } i < 0\},$$
$$\mathrm{Sp}_{\leq 0} = \{\mathrm{X} \in \mathrm{Sp} : \pi_i(X) = 0 \text{ for all } i > 0\}.$$
These two subcategories are called the categories of *connective* and *coconnective* spectra. The heart of spectra, defined as $\mathrm{Sp}^{\heartsuit} = \mathrm{Sp}_{\geq 0} \cap \mathrm{Sp}_{\leq 0}$, is canonically equivalent to the category of abelian groups. Additionally, if $\mathrm{R} \in \mathrm{CAlg}(\mathrm{Sp})$ is a *(commutative) ring spectrum*, then $\pi_*\mathrm{R}$ is a graded ring over the ring $\pi_0\mathrm{R}$.
- There is a symmetric faithful embedding (which is neither essentially surjective nor full):
$$\mathrm{H} : \mathcal{D}(\mathbb{Z}) \to \mathrm{Sp};$$
which is defined on $\mathcal{A}\mathrm{b}$ by taking an abelian group M to the spectrum HM representing singular cohomology with coefficients in M. By this, we mean that if X is a topological space (which we can view as a spectrum in the same way that we can view an abelian group as an object of the derived category), then
$$\mathrm{H}^*(X; \mathrm{M}) \cong [X, \mathrm{HM}],$$
where square brackets denote homotopy classes of maps. Moreover, on homotopy we have $\pi_*\mathrm{HM} = \mathrm{M}$ concentrated in degree 0. This embedding is symmetric monoidal onto its image, meaning
$$\mathrm{H}(\mathrm{R} \otimes_{\mathbb{Z}} \mathrm{R}) \simeq \mathrm{HR} \otimes_{\mathrm{H}\mathbb{Z}} \mathrm{HR}.$$

This last point is what we want to hone in on: we can use the machinery of spectra to calculate derived invariants of classical rings.

1.3. **The plan.** We will begin by generalizing the Picard group to a better behaved homotopical object. Then, we will review the Galois theory of fields and extend this theory to rings and ring spectra. Finally, we will perform computations with various spectral sequences that naturally arise in this framework.

2. Picard groups and Picard spectra

We begin by generalizing the Picard group. A great reference is [MS16].

Definition 2.1. Let $\mathcal{C}$ be a symmetric monoidal category. The *Picard group* of $\mathcal{C}$, denoted $\mathrm{Pic}(\mathcal{C})$, is the group of isomorphism classes of invertible objects; an object $x \in \mathcal{C}$ is *invertible* is there is an object $y \in \mathcal{C}$ such that $x \otimes y \simeq \mathbf{1}$.

We're not going to be worried about size issues here, i.e. we will always have an honest set of isomorphism classes.

Example 2.2. Let $\mathrm{Pic}(\mathrm{A})$ denote $\mathrm{Pic}(\mathrm{Mod}(\mathrm{A}))$ for $\mathrm{A} \in \mathrm{CAlg}(\mathcal{A}\mathrm{b})$ (in other words, A is a commutative ring). This agrees with the classical definition of the Picard group.

Example 2.3. Let $\mathrm{Pic}(X)$ denote $\mathrm{Pic}(\mathrm{QCoh}(\mathcal{O}_X))$ for X a scheme. This agrees with the classical definition of the Picard group.

Example 2.4. Let Sp denote the category of spectra. Then $\mathrm{Pic}(\mathrm{Sp}) \cong \mathbb{Z}$ [HMS94], generated by a suspension of the unit object $\Sigma\mathbb{S}$. The idea behind showing this is to use the Postnikov tower given by the t-structure on Sp.

Example 2.5. Let $\mathcal{D}(\mathbb{Z})$ denote the derived category. Then $\mathrm{Pic}(\mathcal{D}(\mathbb{Z})) \cong \mathbb{Z}$, generated by a suspension of the unit object $\Sigma\mathbb{Z}$. Another way to see this is that $\mathrm{Pic}(\mathcal{D}(\mathbb{Z})) \cong \mathrm{DPic}(\mathbb{Z})$.

We'd like to generalize the Picard group to an invariant which is more computable. One observation is that rather than just remembering the isomorphism classes of invertible objects, we should also remember *how* these objects are isomorphic. This line of thing leads to Picard varieties, Picard schemes, and more generally Picard stacks. We will take a different approach.

Definition 2.6. The *Picard spectrum* of $\mathcal{C}$, denoted $\mathfrak{pic}(\mathcal{C})$, is a connective spectrum encoding the Picard group of $\mathcal{C}$ and higher automorphism data. In particular,

$$\pi_0(\mathfrak{pic}(\mathcal{C})) = \mathrm{Pic}(\mathcal{C}).$$

In other words, the path components, which in this case inherit a group structure, of $\mathfrak{pic}(\mathcal{C})$ are given by the Picard group.

Let $\mathrm{R} \in \mathrm{CAlg}(\mathrm{Sp})$ be a commutative ring spectrum. Then the picard spectrum of R admits a further description:

$$\pi_i(\mathfrak{pic}(\mathrm{R})) = \begin{cases} \mathrm{Pic}(\mathrm{R}) & i = 0 \\ (\pi_0\mathrm{R})^\times & i = 1 \\ \pi_{i-1}\mathrm{R} & i \geq 2 \end{cases}$$

In particular, taking $\mathrm{R} = \mathrm{HA}$ for a commutative ring A, we have

$$\pi_i(\mathfrak{pic}(\mathrm{HA})) = \begin{cases} \mathrm{DPic}(\mathrm{A}) & i = 0 \\ A^\times & i = 1 \\ 0 & i \geq 2 \end{cases}$$

This is the main tool for computing Picard groups in homotopy theory: find some method to understand $\mathfrak{pic}(\mathrm{R})$, then isolate this method at the level of π_0. Our above identification of $\mathrm{Pic}(\mathrm{HA})$ generalizes.

Proposition 2.7. *Let* R *be a connective ring spectrum. Then* $\mathrm{Pic}(\mathrm{R}) \cong \mathrm{DPic}(\pi_*\mathrm{R})$

The above theorem gets us back to the analogy between spectra and the derived category: a connective ring spectrum R should be though of as a sort of "nilpotent thickening" of its underlying homotopy ring $\pi_0\mathrm{R}$, where this thickening is seen in the higher homotopy groups. In the eyes of the Picard group, this is exactly the case.

We will also use the following:

Proposition 2.8. *Let* R *be a ring spectrum such that* $\pi_*\mathrm{R}$ *is a regular noetherian ring. Then*

$$\mathrm{Pic}(\mathrm{R}) \cong \mathrm{Pic}(\pi_0\mathrm{R}) \times \langle \Sigma\mathrm{R} \rangle.$$

3. Galois descent

We bring Galois theory into the story. A good reference for this section is [Rog08].

3.1. **Fields.** Suppose that E is a field with a faithful G-action, for G some finite group, and set $\mathrm{F} = \mathrm{E}^G$ as the fixed points of the action. Then natural inclusion of fixed points $\mathrm{F} \to \mathrm{E}$ is a *G-Galois* extension of fields. In particular, there are two properties we want to highlight.

(1) The induced map $i : \mathrm{F} \to \mathrm{E}^G$ is an isomorphism of fields.
(2) The map of rings
$$h : \mathrm{E} \otimes_{\mathrm{F}} \mathrm{E} \to \prod_G \mathrm{E}$$
given by $e_1 \otimes e_2 \mapsto \{e_1 \cdot ge_2 : g \in G\}$ is an isomorphism of commutative rings.

We can ask about descent along Galois extensions of fields. If $\mathrm{F} \to \mathrm{E}$ is G-Galois and $\mathrm{V} \in \mathrm{Vect}(\mathrm{E})$, can we descend V to some F-vector space? That is, is there some $\mathrm{W} \in \mathrm{Vect}(\mathrm{F})$ such that $\mathrm{W} \otimes_{\mathrm{F}} \mathrm{E} \cong \mathrm{V}$? This is kinda trivial to ask here; everything is free, so we can just choose a basis $\{e_1, \ldots e_n\}$ for V. The fixed points V^G are a vector space over $\mathrm{E}^G = \mathrm{F}$ of dimension n on the same basis, and clearly when we base change back to E, we get our original vector space V. Another way to say this is that there is an equivalence of categories
$$\mathrm{Vect}(\mathrm{E})^G \cong \mathrm{Vect}(\mathrm{F}).$$

Example 3.1. There is a C_2-Galois extension of fields
$$\mathbb{Q} \to \mathbb{Q}(i),$$
where the Galois action is determined by $i \mapsto -i$. If V is any vector space over $\mathbb{Q}(i)$, then the fixed points of the conjugation action give a vector space of the same dimension over $\mathbb{Q}$.

3.2. **Rings.** We can take the two properties we isolated from the definition of a Galois extension of fields and generalize them. A ring homomorphism $\mathrm{A} \to \mathrm{B}$ of commutative rings is G-Galois if G acts on B through A-algebra maps such that the maps
$$i : \mathrm{A} \to \mathrm{B}^G,$$
$$h : \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \prod_G \mathrm{B}$$
are isomorphisms.

Example 3.2. If $\mathrm{F} \to \mathrm{E}$ is a G-Galois extension of number fields, then $\mathcal{O}_{\mathrm{F}} \to \mathcal{O}_{\mathrm{E}}$ is a G-Galois extension of rings of integers.

We can ask about Galois descent again. Thinks might seem hard, but we should remember something mentioned in Jay's 1-2-3 talk: a ring map $\mathrm{A} \to \mathrm{B}$ is descendable if and only if it is faithfully flat.

Proposition 3.3. *If $\mathrm{A} \to \mathrm{B}$ is a G-Galois extension of commutative rings, then it is faithfully flat.*

Proof. It is a little tedious to show, but is true that B is finitely generated and projective over A.

Idea:

- Use the Galois trace $\mathrm{tr} : \mathrm{B} \to \mathrm{A}$ given by
$$\mathrm{tr}(b) = \sum_{g \in G} g \cdot b.$$
Note that this is Galois invariant, so we do actually land in A.
- Surjectivity of $h : \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \prod_G \mathrm{B}$ implies that there must exist elements $x_1, \ldots x_n, y_1, \ldots y_n$, where $n = |G|$, such that
$$\sum_{i=1}^{n} x_i (g \cdot y_i) = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases}$$

- Define a function $\varphi_i : \mathrm{B} \to \mathrm{A}$ by
$$\varphi_i(b) = \mathrm{tr}(b \cdot y_i).$$
We can now rewrite any element of B as
$$b = \sum_{i=1}^{n} x_i \cdot \varphi_i(b),$$
which in particular gives a dual basis for B as an A-module, hence gives projectivity. Finite generation is immediate from finiteness of G.

Projective modules are flat, so it remains to show that B is faithful, i.e. that $ab \neq 0$ for $a \neq 0$. Better yet, Nakayama's lemma tells us that we must only show that $\mathrm{B}_{\mathfrak{m}} \neq 0$ for all maximal ideals $\mathfrak{m} \subseteq \mathrm{A}$. This is immediate: localization is exact, and $\mathrm{A} \to \mathrm{B}$ is injective, hence $\mathrm{A}_{\mathfrak{m}} \to \mathrm{B}_{\mathfrak{m}}$ is also injective. Since $\mathrm{A}_{\mathfrak{m}} \neq 0$, we're done. $\square$

As before, another way that we can state this descent property is that there is an equivalence of categories
$$\mathrm{Mod}(\mathrm{B})^G \cong \mathrm{Mod}(\mathrm{A}).$$

Example 3.4. Let's continue our example from before: the C_2-Galois extension of fields
$$\mathbb{Q} \to \mathbb{Q}(i)$$
gives a C_2-Galois extension on rings of integers
$$\mathbb{Z} \to \mathbb{Z}[i].$$
Again, we can perform descent along this map, which is given by the conjugation action on $\mathbb{Z}[i]$.

Example 3.5. We can also form examples which do not come from the Galois theory of fields. Here is a trivial example. Take any ring A with a trivial G-action. We can form the product $\mathrm{B} = \prod_G \mathrm{A}$, where G acts through the index. The fixed point of the action is isomorphic to the diagonal, which is isomorphic to A. It is also not hard to show that the twised product map $h : \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \prod_G \mathrm{B}$ is an isomorphism, practically by construction. This gives a G-Galois extension
$$A \to \prod_G A.$$

3.3. **Ring spectra.** Again, we will take this to spectra. If $\mathrm{R} \to \mathrm{S}$ is a map of ring spectra, where G acts on S through A-algebra maps, then this map is G-Galois if the maps
$$i : \mathrm{R} \to \mathrm{S}^{hG},$$
$$h : \mathrm{B} \otimes_A \mathrm{B} \to \prod_G B$$
are equivalences of ring spectra. Here, $(-)^{hG}$ denotes the homotopy fixed points, a detail which is not very important here.

Example 3.6. The map $\mathrm{A} \to \mathrm{B}$ is a G-Galois extension of commutative rings if and only if $\mathrm{HA} \to \mathrm{HB}$ is a G-Galois extension of ring spectra. The idea here is to use a few useful spectral sequences. In particular, there is a *homotopy fixed point spectral sequence*
$$E_2^{s,f} = \mathrm{H}^f(G, \pi_{s+f}\mathrm{HB}) \implies \pi_s(\mathrm{HB}^{hG}), \quad d_r : E_r^{s,f} \to E_r^{s-1,f+r},$$
and a *Künneth spectral sequence*
$$E^2_{s,f} = \mathrm{Tor}^{\pi_*\mathrm{HR}}_{f,s+f}(\pi_*\mathrm{HT}, \pi_*\mathrm{HT}) \implies \pi_s(\mathrm{HT} \otimes_{\mathrm{HR}} \mathrm{HR}).$$
One can show that these spectral sequences collapse, giving the result.

To continue our running example, there is a C_2-Galois extension of ring spectra
$$\mathrm{H}\mathbb{Z} \to \mathrm{H}\mathbb{Z}[i].$$

Example 3.7. The cohomology theories of complex and real topological K-theory give rise to ring spectra $\mathrm{KU}, \mathrm{KO} \in \mathrm{Sp}$. Bott periodicity determines the homotopy groups of these spectra:

$$\pi_*\mathrm{KU} = \mathbb{Z}[u^{\pm 1}] = \begin{cases} \mathbb{Z} & * = 2k \\ 0 & * = 2k+1 \end{cases},$$

$$\pi_*\mathrm{KO} = \mathbb{Z}[\eta, a, b]/(2\eta, \eta^3, \eta a, a^2 = 4b) \begin{cases} \mathbb{Z} & * = 4k \\ \mathbb{Z}/2 & * = 8k+1, 8k+2 \\ 0 & else \end{cases}.$$

In particular, there are equivalences of spectra $\mathrm{KO} \simeq \Sigma^8\mathrm{KO}$ and $\mathrm{KU} \simeq \Sigma^2\mathrm{KU}$. There is a complexification map $\mathrm{KO} \to \mathrm{KU}$ which is a C_2-Galois extension of ring spectra. The C_2-action on KU comes from complex conjugation, and is determined on the level of homotopy by $u \mapsto -u$.

Example 3.8. I won't talk about this example, but it is too important to omit. The unit object in any symmetric monoidal category is a ring object. In particular, the sphere spectrum $\mathbb{S}$ is a ring object in Sp. Its homotopy groups $\pi_*\mathbb{S}$ are the stable homotopy groups of spheres, and so naturally we want to find a way to calculate these things. One such approach is to divide and conquer in an organized and meaningful way: this is the goal of chromatic homotopy theory. First, since $\pi_0\mathbb{S} \cong \mathbb{Z}$ (which we can see unstably by the isomorphism $\pi_n S^n \cong \mathbb{Z}$) and all the higher homotopy groups are finite abelian groups by a theorem of Serre, we can try to calculate the p-completed ring $\pi_*\mathbb{S}_p$, the homotopy groups of the p-completed sphere spectrum.

It turns out that there are a wealth of Galois extensions of the this sphere if we "invert homotopical primes". To be slightly more precise, there are a family of ring spectra at each prime p called the Morava K-theories, denoted $\mathrm{K}(n)$ for $n \geq 0$. These ring spectra have homotopy groups given

$$\pi_*\mathrm{K}(n) \cong \mathbb{F}_p[v_n^{\pm 1}],$$

where $|v_n| = 2(p^n - 1)$. By convention, we set $\mathrm{K}(0) = \mathrm{H}\mathbb{Q}$ and $\mathrm{K}(\infty) = \mathrm{H}\mathbb{F}_p$, which we can see on homotopy groups. These v_n's act as higher primes in the category of spectra, and so a useful thing to do in our "divide and conquer" idea is to work one v_n at a time: that is, to compute the homotopy groups of the $\mathrm{K}(n)$-local sphere $\mathrm{L}_{\mathrm{K}(n)}\mathbb{S}_p$. A foundational result is that there is a profinite Galois extension of this local sphere:

$$\mathrm{L}_{\mathrm{K}(n)}\mathbb{S}_p \to \mathrm{E}_n,$$

with Galois group $\mathbb{G}_n$ known as the Morava stabilizer group. The ring spectrum E_n is the Morava E-theory or Lubin-Tate spectrum, and is related to the Lubin-Tate space of lifts. These ideas, originally of Morava, are the start of a beautiful connection between chromatic homotopy theory and arithmetic geometry. In this light, one should interpret E_n as the algebraic closure of the local sphere. When $n = 1$, we have models for many of these characters given in terms of K-theory: $\mathrm{K}(1) = \mathrm{KU}/p$, $\mathrm{E}_1 = \mathrm{KU}_p$, and $\mathbb{G}_1 = \mathbb{Z}_p^\times$ are the Adams operations on K-theory. The descent spectral sequence discussed below for the Galois extension

$$\mathrm{L}_{\mathrm{KU}/p}\mathbb{S}_p \to \mathrm{KU}_p$$

is essentially determined by three interesting facts:

(1) Adams's study of the J-homomorphism and vector fields on spheres;
(2) Quillen's determination of the algebraic K-theory of finite fields, and;
(3) Suslin's proof that $\mathrm{K}(\mathbb{C})_p \simeq \mathrm{ku}_p$, where $\mathrm{K}(\mathbb{C})$ denotes the algebraic K-theory spectrum for the complex numbers and $\mathrm{ku} = \tau_{\geq 0}\mathrm{KU}$ is the connective cover of complex K-theory.

For $p = 2$, the Galois group $\mathbb{G}_1 = \mathbb{Z}_2^\times \cong \mathbb{Z}_2 \times C_2$ contains a copy of the cyclic gorup of order 2. If we take homotopy fixed points with respect to this subgroup, we recover the above example (completed at the prime 2): there is a C_2-Galois extension

$$\mathrm{KO}_2 \to \mathrm{KU}_2.$$

For $n \geq 2$, the Galois extension $\mathbb{S}_p \to \mathrm{E}_n$ is very interesting and very complicated. I will refrain from saying anything else, other than there are many meaningful ways to talk about these topics in terms of the moduli stack of formal groups.

We will now investigate Galois descent in this context. As we saw above, a useful tool for calculating homotopy groups along a Galois extension is the homotopy fixed point spectral sequence. Our framework with the Picard spectrum gives us even more, which we outline now. Suppose $\mathrm{R} \to \mathrm{S}$ is a G-Galois extension of ring spectra. The action of G on S can be lifted to the module category Mod(S) using the isomorphism $\mathrm{M} \simeq \mathrm{M} \otimes_{\mathrm{S}} \mathrm{S}$. In fact, every Galois extension of ring spectra (where the G-action is faithful) is faithfully flat. Thus we still have descent, which we'll witness as an equivalence of categories

$$\mathrm{Mod(R)} \simeq \mathrm{Mod(S)}^{hG}.$$

This equivalence is actually symmetric monoidal,which leads to an equivalence of Picard spectra:

$$\mathfrak{pic}(\mathrm{R}) \simeq \tau_{\geq 0}\mathfrak{pic}(\mathrm{S})^{hG}.$$

We must truncate on the right side, as taking fixed points of a connective spectrum is not necessarily connective. This gives a descent spectral sequence

$$E_2^{s,f} = \mathrm{H}^f(G, \pi_{s+f}(\mathfrak{pic}(\mathrm{S}))) \implies \pi_s(\mathfrak{pic}(\mathrm{S})^{hG}),$$

whose abutment at $s = 0$ is the Picard group Pic(R). Moreover, we saw that for $s + f \geq 2$, there is an isomorphism of groups

$$\mathrm{H}^f(G, \pi_{s+f}\mathrm{S}) \cong \mathrm{H}^f(G, \pi_{s+f}(\mathfrak{pic}(\mathrm{S})).$$

This isomorphism actually extends to differentials in the spectral sequence, meaning that for $s+f \geq 2$, the differentials in the descent spectral sequence for the Picard group are isomorphic to the differentials in the descent spectral sequence for the original G-Galois extension.

Remark 3.9. Let $\mathrm{F} \to \mathrm{E}$ be a G-Galois extension of fields. This gives a G-Galois extension $\mathcal{O}_{\mathrm{F}} \to \mathcal{O}_{\mathrm{E}}$ of rings of integers, hence a G-Galois extension $\mathrm{H}\mathcal{O}_{\mathrm{F}} \to \mathrm{H}\mathcal{O}_{\mathrm{E}}$ of ring spectra. To this we have a homotopy fixed point spectral sequence

$$\mathrm{H}^f(G, \pi_{s+f}\mathrm{H}\mathcal{O}_E) \implies \pi_s\mathrm{H}\mathcal{O}_{\mathrm{F}}$$

which is kinda silly, since we already know the homotopy groups of any Eilenberg-Maclane spectrum. However, this does give a descent spectral sequence on Picard spectra

$$\mathrm{H}^f(G, \pi_{s+f}(\mathfrak{pic}(\mathrm{H}\mathcal{O}_{\mathrm{E}}))) \implies \pi_s(\mathfrak{pic}(\mathcal{O}_{\mathrm{E}})^{hG}),$$

whose abutment at $s = 0$ is the derived Picard group $\mathrm{DPic}(\mathcal{O}_{\mathrm{F}})$. This gives a really cool new way to compute the Picard group of classical rings!

4. Computations

We're going to try and compute with this spectral sequence. First, let's take a look at a homotopical lift of the classical Picard group.

Example 4.1. Take the C_2-Galois extension of fields that we've been running with

$$\mathbb{Q} \to \mathbb{Q}(i).$$

This gives a C_2-Galois extension of rings

$$\mathbb{Z} \to \mathbb{Z}[i],$$

hence a C_2-Galois extension of ring spectra

$$\mathrm{H}\mathbb{Z} \to \mathrm{H}\mathbb{Z}[i].$$

We can try to use the descent spectral sequence to compute $\mathrm{DPic}(\mathbb{Z}[i])$. This spectral sequence takes the form

$$\mathrm{H}^f(C_2, \pi_{s+f}(\mathfrak{pic}(\mathrm{H}\mathbb{Z}[i]))) \implies \pi_s(\mathfrak{pic}(\mathrm{H}\mathbb{Z}[i])^{C_2}).$$

On the surface, this doesn't look so bad. We already know the homotopy groups of the target:

$$\pi_i \mathfrak{pic}(\mathrm{H}\mathbb{Z}) = \begin{cases} \mathrm{DPic}(\mathbb{Z}) \cong \mathbb{Z} & i = 0 \\ (\pi_0 \mathrm{H}\mathbb{Z})^\times \cong \mathbb{Z}^\times \cong C_2 & i = 1 \\ 0 & i \geq 2 \end{cases}$$

And we can figure out the homotopy groups of the coefficients, at least up to the Picard group of interest:

$$\pi_i \mathfrak{pic}(\mathrm{H}\mathbb{Z}[i]) = \begin{cases} \mathrm{DPic}(\mathbb{Z}[i]) & i = 0 \\ (\pi_0 \mathrm{H}\mathbb{Z}[i])^\times \cong \mathbb{Z}[i]^\times \cong C_2 \times C_2 & i = 1 \\ 0 & i \geq 2 \end{cases}$$

This deduction of the coefficient tells us something nice: when $s + f \geq 2$, the term $E_2^{s,f}$ vanishes. Moreover, since the E_2-page is given by group cohomology, we already know that $E_2^{s,f}$ vanishes for $f < 0$. This gives us a lot of vanishing! Let's see what the E_2-page looks like: Again, we are only

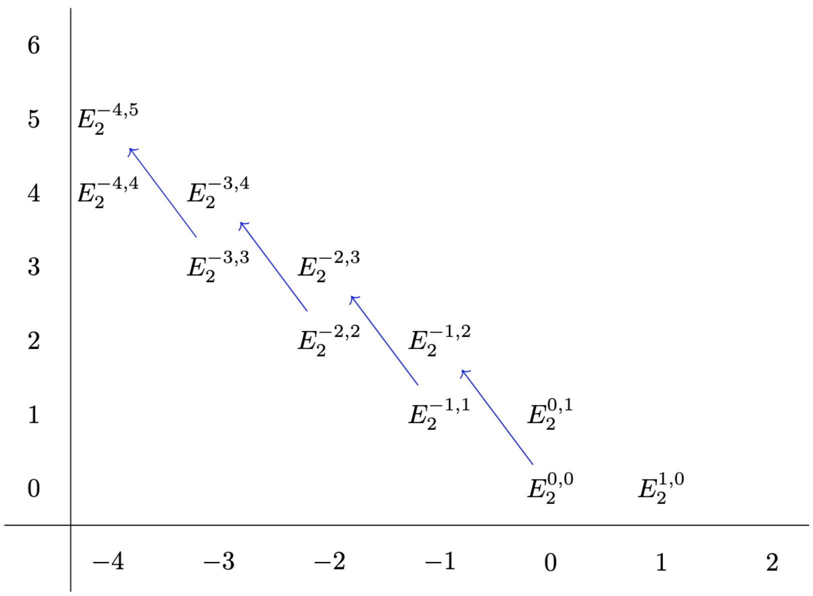

Figure 1. The E_2-page of the descent spectral sequence for $\pi_* \mathfrak{pic}(\mathrm{H}\mathbb{Z}[i])$, with d_2-differentials in blue.

concerned with the abutment at $s = 0$, meaning the 0th column of this spectral sequence. For degree reasons, we have $E_3 = E_\infty$. Notice that the term $E_2^{0,1} = \mathrm{H}^1(C_2; C_2 \times C_2)$ is already stable, as there are no possible differentials. There is a potential differential to analyze:

$$d_2 : E_2^{0,0} \cong \mathrm{H}^0(C_2, \mathrm{DPic}(\mathbb{Z}[i])) \to \mathrm{H}^2(C_2, C_2 \times C_2) = E_2^{-1,2},$$

but now we're getting into some nasty group cohomology computations just to solve for the coefficients of a cohomology group. It's not the simplest, and in fact (as I found out while preparing this example), it isn't consistent in a precise way. We could run the same argument for the C_2-Galois extension

$$\mathbb{Q} \to \mathbb{Q}(\sqrt{d})$$

for any non square in $\mathbb{Z}$, and the above computation seems to imply that since the E_2-page will look the same, meaning concentrated in two bands, and the C_2-action on $\mathbb{Z}[\sqrt{d}]^\times \cong C_2 \times C_2$ is still given by negating the second factor, the answer actually *depends* on d! The classical Picard group of these input rings are the ideal class groups, and these can be viewed as a measure of the failure of a ring to be a UFD. For example, you can use actualy number theory to show that $\mathbb{Z}[i], \mathbb{Z}[\sqrt{2}], \mathbb{Z}[\sqrt{3}]$, and $\mathbb{Z}[\sqrt{5}]$ all have trivial Picard group, but $\mathbb{Z}[\sqrt{82}]$ and $\mathbb{Z}[\sqrt{-5}]$ have Picard group $\mathbb{Z}/2$. Wacky!

Let's compute an interesting example to me, as a homotopy theorist!

Example 4.2. Now, we compute the Picard group of KU and KO.

Proposition 4.3. $\mathrm{Pic}(\mathrm{KU}) \cong \mathbb{Z}/2$.

Proof. The homotopy ring of KU is given by $\pi_*\mathrm{KU} = \mathbb{Z}[u^{\pm 1}]$, where $|u| = 2$. Since this is a regular noetherian ring, we have that Pic(KU) is generated by $\Sigma\mathrm{KU}$. In particular, $\mathrm{Pic}(\pi_0\mathrm{KU}) = \mathrm{Pic}(\mathbb{Z}) = 0$ and $\Sigma^2\mathrm{KU} \simeq \mathrm{KU}$, giving the result. □

Theorem 4.4. $\mathrm{Pic}(\mathrm{KO}) \cong \mathbb{Z}/8$.

Proof. First, we recall the homotopy fixed point spectral sequence for the Galois extension KO → KU. This has signature

$$\mathrm{H}^f(C_2, \pi_{s+f}\mathrm{KU}) \implies \pi_s\mathrm{KO}.$$

For degree reasons, there is no differential until the E_3-page, which we present below. This potential

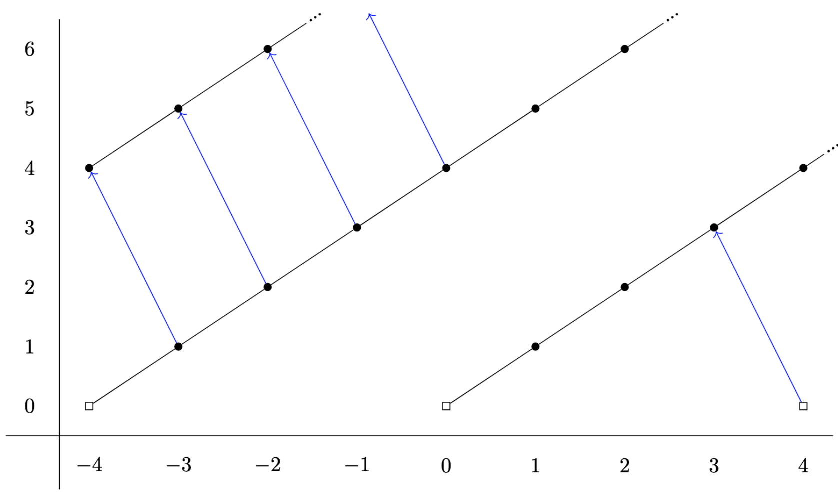

FIGURE 2. The E_3-page of the HFPSS(KO), with d_3-differentials in blue. A □ denotes a $\mathbb{Z}$ and a • denotes a $\mathbb{Z}/2$. A black diagonal line represents multiplication by the nonzero element in $\mathrm{H}^1(C_2, \mathbb{Z}_\sigma)$.

differential $d_3 : E_3^{4,0} \to E_3^{3,3}$ can be deduced by knowing the answer already. If $\pi_3\mathrm{KO} = 0$, then there must be a differential killing the class in $E_3^{3,3}$, and so the d_3-differential is nonzero. Similarly, there will be a differential $d_7 : E_7^{4,0} \to E_7^{3,7}$, and so on.

Now, we use the descent spectral sequence:

$$\mathrm{H}^f(C_2, \pi_{s+f}\mathfrak{pic}(\mathrm{KU})) \implies \pi_s\mathfrak{pic}(\mathrm{KU})^{hC_2},$$

whose abutment at $s = 0$ is the answer we desire. But recall that for $s+f \geq 2$, this spectral sequences is isomorphic to a shifted version of the previous HFPSS! Here is a picture of the E_3-page.

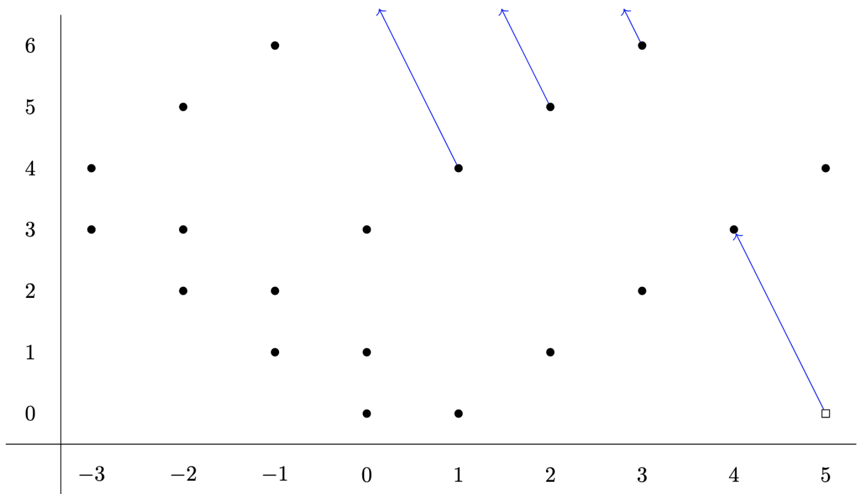

FIGURE 3. The E_3-page of the descent spectral sequence for $\pi_*\mathfrak{pic}(\mathrm{KO})$.

Notice that to the left of $s+f=0$, there are no groups due to the connectivity of the Picard spectrum. Moreover, since $\pi_1\mathfrak{pic}(\mathrm{KU}) = \mathbb{Z}^\times = C_2$, we can calculate the entire $s+f=1$-diagonal as all $\mathbb{Z}/2$'s. Then, in the region $s+f \geq 2$, we can import the differentials from the HFPSS. The only question in the 0-column, then, are those three isolated dots. However, Bott periodicity tells us that $\mathrm{KO} \simeq \Sigma^8\mathrm{KO}$. In particular, this gives us a lower bound of $\mathbb{Z}/8$ in the Picard group, since $\Sigma^i\mathrm{KO} \neq \Sigma^j\mathrm{KO}$ for $i \neq j \bmod 8$ (which we can see on homotopy groups. Thus, those three dots must all survive to the E_∞-page, and our lower bound forces the hidden extensions, giving the result. $\square$

As a last remark, we note that it is simple to calculate the Picard groups of the connective covers ku and ko: they are both $\mathbb{Z}$. However, it is *not* true that there is a C_2-Galois extension $\mathrm{ko} \to \mathrm{ku}$. One can see this purely by analyzing the homotopy fixed point spectral sequence for the C_2-action on ku: there will be infinitely many $\mathbb{Z}/2$'s in negative stem degrees which are not the target of any differentials, hence survive to $\pi_*\mathrm{ku}^{hC_2}$, while $\pi_*\mathrm{ko} = 0$ for $* < 0$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON
Email address: jackmann@uw.edu