# MATH 480: HOMOTOPY THEORY HOMEWORK 1

ABSTRACT. Homework 1 due in class on **Wednesday, April 8**.

## PROBLEM 1

What is your favorite topological space? Tell me something fun about it. Why is it you favorite? Does it have a nice property? Does it look cool, and can you include a drawing or image of it? Do you know any of its homotopy groups?

**ANSWER:**

My favorite topological space is the space $\mathbb{RP}^\infty$, the infinite dimensional real projective space. The finite dimensional space $\mathbb{RP}^n$is defined to be the space whose points are represented by lines going through the origin in $\mathbb{R}^{n+1}$. For example, one can show that $\mathbb{RP}^1 \cong S^1$. Another way to form $\mathbb{RP}^n$ is as the quotient of $\mathbb{R}^{n+1} - \{0\}$ where we identify any point $x \in \mathbb{R}^{n+1} - \{0\}$ with $\lambda \cdot x$ for any $\lambda \in \mathbb{R}$.

One can define $\mathbb{RP}^\infty$ as the colimit of the natural inclusions

$$\mathbb{RP}^1 \to \mathbb{RP}^2 \to \mathbb{RP}^3 \to \cdots$$

This space has many nice properties.

- The homotopy groups of $\mathbb{RP}^\infty$ are very simple:
$$\pi_i(\mathbb{RP}^\infty) = \begin{cases} \mathbb{Z}/2 & i = 1 \\ 0 & \text{else.} \end{cases}$$
- There is a surjective map $S^\infty \to \mathbb{RP}^\infty$ from the infinite dimensional sphere which we may interpret as identifying antipodal points.
- One may represent this space as a CW complex (which we will talk about later in the course) with a single cell in each dimension. The attaching maps for these cells play a key role in homotopy theory.
- One may view $\mathbb{RP}^\infty$ as the classifying space BO(1). What this means is that there is a one-to-one correspondence between certain covering maps called principal $C_2$-bundles $E \to X$ over a space $X$ with homotopy classes of maps $X \to \mathbb{RP}^\infty$.

## PROBLEM 2 (**SQUARE LEMMA**)

Let $X$ be a topological space, and let $F : I \times I \to X$ be a continuous function. Define the following paths in $X$:

- $f : I \to X$ defined by $f(t) = F(t, 0)$;
- $g : I \to X$ defined by $g(t) = F(1, t)$;
- $h : I \to X$ defined by $h(t) = F(0, t)$;
- $k : I \to X$ defined by $k(t) = F(t, 1)$.

Show that there is a homotopy $g \cdot f \simeq k \cdot h$, where $\cdot$ denotes the path product. (**Hint:** Make some dummy space $X$, draw what the image of $F$ looks like in $X$, and try to identify $f, g, h, k$ in terms of $F$.)

**ANSWER:**

Let's set up what we have and what we need. The path product $g \cdot f : I \to X$ is given by

$$g \cdot f(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1, \end{cases}$$

and the path product $k \cdot h : I \to X$ is given by

$$k \cdot h(t) = \begin{cases} h(2t) & 0 \leq t \leq 1/2 \\ k(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

To show that $g \cdot f \simeq k \cdot h$, we want a homotopy

$$H : I \times I \to X,$$

where $H(s, 0) = g \cdot f(s)$ and $H(s, 1) = k \cdot h(s)$.

The way to do this is to first construct a homotopy from the edges of the square, completely internal to $I \times I$, then compose with $F$. It is important to work this way because it is often tempting, but is not actually correct, to try to write something such as $tF(x, y) + (1 - t)F(x, y)$. This symbol actually doesn't make any sense! Since $F : I \times I \to X$, the element $F(x, y) \in X$ lives in a topological space. In this space, we do not have any algebraic structure, and so it does not make sense to scale by some constant.

So, consider the path $\gamma : I \to I \times I$ given by

$$\gamma(t) = \begin{cases} (2t, 0) & 0 \leq t \leq 1/2 \\ (1, 2t-1) & 1/2 \leq t \leq 1, \end{cases}$$

and consider the path $\varphi : I \to I \times I$ given by

$$\varphi(t) = \begin{cases} (0, 2t) & 0 \leq t \leq 1/2 \\ (2t-1, 1) & 1/2 \leq t \leq 1. \end{cases}$$

There is a homotopy $K : I \times I \to I \times I$ from $\gamma$ to $\varphi$ given by

$$K(s, t) = (1 - t)\gamma(s) + t\varphi(s).$$

This is the homotopy which goes in a straight line from $\gamma$ to $\varphi$. The composition

$$H = F \circ K : I \times I \to X$$

is the desired homotopy, as $H(s, 0) = F(K(s, 0)) = F(\gamma(s)) = g \cdot f$, and $H(s, 1) = F(K(s, 1)) = F(\varphi(s)) = k \cdot h$.

## PROBLEM 3 (**FUNCTORIALITY OF FUNDAMENTAL GROUP**)

Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be maps of based spaces (recall that this is just a continuous map of topological spaces which sends basepoint to basepoint).

(a) Show that the function

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad f_*([\gamma]) = [f \circ \gamma]$$

is a group homomorphism.

(b) Show that for any $[\gamma] \in \pi_1(X, x_0)$, we have

$$(g_* \circ f_*)([\gamma]) = (g \circ f)_*([\gamma]).$$

(c) Show that $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$. In other words, the identity map on $X$ induces the identity map on $\pi_1(X, x_0)$.

**ANSWER:**

These are pretty plug-and-chug computations, but they are important. For part (a), suppose that $[\gamma], [\varphi] \in \pi_1(X, x_0)$ are any loops. We must show that

$$f_*([\gamma] \cdot [\varphi]) = f_*([\gamma]) \cdot f_*([\varphi]).$$

We can just use the definitions.

$$f_*([\gamma] \cdot [\varphi]) = f_*([\gamma \cdot \varphi]) = [f \circ (\gamma \cdot \varphi)] = [(f \circ \gamma) \cdot (f \circ \varphi)] = [f \circ \gamma] \cdot [f \circ \varphi] = f_*([\gamma]) \cdot f_*([\varphi]).$$

For part (b), observe that

$$(g_* \circ f_*)([\gamma]) = g_*(f_*([\gamma])) = g_*([f \circ \gamma]) = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma]).$$

Finally, for part (c), observe that

$$(\mathrm{id}_X)_*([\gamma]) = [\mathrm{id}_X \circ \gamma] = [\gamma] = \mathrm{id}_{\pi_1(X,x_0)}([\gamma]).$$

## PROBLEM 4

A *topological group* is a topological space $G$ equipped with continuous maps

$$m : G \times G \to G \quad \text{(multiplication)},$$

$$i : G \to G \quad \text{(inverse)},$$

such that the diagrams

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} & G \times G \arrow[d, "m"] \\ G \arrow[ur, "{(\mathrm{id}_G, i)}"] \arrow[r, "\mathrm{id}_G"] & G \end{tikzcd}$$

commute. Let $e = m(g, i(g))$ for any $g \in G$ be the identity. Show that the fundamental group $\pi_1(G, e)$ is abelian by doing the following: for any two elements $\alpha, \beta \in \pi_1(G, e)$, show that the product $\alpha \cdot \beta \in \pi_1(G, e)$ is homotopic to both the map

$$f_1 : I \to G, \quad f_1(t) = m(\alpha(t), \beta(t))$$

and to the map

$$f_2 : I \to G, \quad f_2(t) = m(\beta(t), \alpha(t)).$$

Notice that we are **NOT** assuming that $G$ is abelian.

**ANSWER:**

This one is a little wild! There are a lot of ways to prove that this is true. I am going to present one which uses the square lemma. Consider the map $F : I \times I \to G$ given by $F(s, t) = m(\alpha(s), \beta(t))$, where $\alpha, \beta : I \to G$ are any two loops based at the identity. This is certainly continuous, as the component functions are continuous as well as the multiplication on $G$. Notice as well that $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = e$ are all the identity element of the group $G$. Now, we have edge paths of our square:

$$F(s, 0) = m(\alpha(s), \beta(0)) = m(\alpha(s), e) = \alpha(s)$$
$$F(1, t) = m(\alpha(1), \beta(t)) = m(e, \beta(t)) = \beta(t)$$
$$F(0, t) = m(\alpha(0), \beta(t)) = m(e, \beta(t) = \beta(t)$$
$$F(s, 1) = m(\alpha(s), \beta(1)) = m(\alpha(s), e) = \alpha(s).$$

Thus, the square lemma implies that $\alpha \cdot \beta \simeq \beta \cdot \alpha$, proving the claim.

## PROBLEM 5

Let $M = I \times I/((t,0) \sim (1-t,1))$ be the Möbius strip. Show that $M$ is homotopy equivalent to $S^1$.

**ANSWER:**

Notice that there are many subspaces of $M$ which are homeomorphic to $S^1$. Letting $q : I \times I \to M$ denote the quotient map, consider the subspace $q(\{1/2\} \times I) \subseteq M$. Since $1/2 = 1 - 1/2$, the strip $\{1/2\} \times I$ is identified with $S^1$ in the quotient space. In other words, the restriction $q|_{\{1/2\}\times I} : \{1/2\} \times I \to q(\{1/2\} \times I)$ is homeomorphic to the usual quotient map $I \to S^1$.

There is an inclusion map $i : S^1 \to M$ sending any point $x \in S^1$ to $(1/2, x) \in q(\{1/2\} \times I) \subseteq M$. There is also a surjective map $p : M \to q(\{1/2\} \times I) \cong S^1$ which collapses onto the central circle of the Möbius strip. More precisely,

$$p(s,t) = q(1/2,t),$$

i.e. we send every point on the Möbius strip to the corresponding point on the central circle and compose with the restriction isomorphism $q|_{\{1/2\}\times I}$. Observe that then

$$p \circ i(x) = p(1/2, x) = x = \mathrm{id}_{S^1}.$$

Thus the continuous map

$$H_1 : S^1 \times I \to S^1, \quad H(x,t) = p \circ i(x)$$

is a (silly) homotopy from $p \circ i$ to $\mathrm{id}_{S^1}$.

Now, observe that

$$i \circ p(x,y) = i(y) = (1/2, y).$$

Consider the map

$$H_2 : M \times I \to M, \quad H_2((x,y),t) = ((1-t)x + t(1/2), y).$$

Then $H_2((x,y),0) = (x,y)$ and $H_2((x,y),1) = (1/2,y)$. Thus we also have a homotopy from $i \circ p$ to $\mathrm{id}_M$, so we have produced a homotopy equivalence $M \simeq S^1$.