

MATH 480: HOMOTOPY THEORY HOMEWORK 1

ABSTRACT. Homework 1 due in class on **Wednesday, April 8.**

1. PROBLEMS

- (1) What is your favorite topological space? Tell me something fun about it. Why is it you favorite? Does it have a nice property? Does it look cool, and can you include a drawing or image of it? Do you know any of its homotopy groups?
- (2) (**Square Lemma**) Let X be a topological space, and let $F : I \times I \rightarrow X$ be a continuous function. Define the following paths in X :
- $f : I \rightarrow X$ defined by $f(t) = F(t, 0)$;
 - $g : I \rightarrow X$ defined by $g(t) = F(1, t)$;
 - $h : I \rightarrow X$ defined by $h(t) = F(0, t)$;
 - $k : I \rightarrow X$ defined by $F(t) = k(t, 1)$.

Show that there is a homotopy $g \cdot f \simeq k \cdot h$, where \cdot denotes the path product. (**Hint:** Make some dummy space X , draw what the image of F looks like in X , and try to identify f, g, h, k in terms of F .)

- (3) (**Functoriality of fundamental group**) Let $f : (X, x_0) \rightarrow (Y, y_0)$ and $g : (Y, y_0) \rightarrow (Z, z_0)$ be maps of based spaces (recall that this is just a continuous map of topological spaces which sends basepoint to basepoint).

(a) Show that the function

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0), \quad f_*([\gamma]) = [f \circ \gamma]$$

is a group homomorphism.

(b) Show that for any $[\gamma] \in \pi_1(X, x_0)$, we have

$$(g_* \circ f_*)([\gamma]) = (g \circ f)_*([\gamma]).$$

(c) Show that $(\text{id}_X)_* = \text{id}_{\pi_1(X, x_0)}$. In other words, the identity map on X induces the identity map on $\pi_1(X, x_0)$.

- (4) A *topological group* is a topological space G equipped with continuous maps

$$m : G \times G \rightarrow G \quad (\text{multiplication}),$$

$$i : G \rightarrow G \quad (\text{inverse}),$$

such that the diagrams

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \text{id}_G} & G \times G \\ \text{id}_G \times m \downarrow & & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array} \qquad \begin{array}{ccc} & & G \times G \\ (\text{id}_G, i) \nearrow & & \downarrow m \\ G & \xrightarrow{\text{id}_G} & G \end{array}$$

commute. Let $e = m(g, i(g))$ for any $g \in G$ be the identity. Show that the fundamental group $\pi_1(G, e)$ is abelian by doing the following: for any two elements $\alpha, \beta \in \pi_1(G, e)$, show that the product $\alpha \cdot \beta \in \pi_1(G, e)$ is homotopic to both the map

$$f_1 : I \rightarrow G, \quad f_1(t) = m(\alpha(t), \beta(t))$$

and to the map

$$f_2 : I \rightarrow G, \quad f_2(t) = m(\beta(t), \alpha(t)).$$

Notice that we are **NOT** assuming that G is abelian.

- (5) Let $M = I \times I / ((t, 0) \sim (1 - t, 1))$ be the mobius strip. Show that M is homotopy equivalent to S^1 .