# MATH 480: HOMOTOPY THEORY

JACKSON MORRIS

## Contents

1. Motivation/Background 2
Lecture 1 2
Lecture 2 4
2. The Fundamental Group 5
Lecture 3 5
Lecture 4 6
Lecture 5 8

## 1. MOTIVATION/BACKGROUND

**Lecture 1.**

The things that mathematicians care about can often be distinguished by two qualities: *existence* and *uniqueness*. That is, suppose a mathematician thinks of some concept $X$. There are two obvious questions to ask:

- Does such a concept $X$ actually exist? If so, can we explicitly construct it?
- If a concept $X$ exists, is it unique? If it isn't unique, can we classify all such types of concepts $X$?

In this course, we will be studying topological spaces. Recall that a *topological space* is a set $X$ together with a collection $\mathcal{T}$ of subsets, called the *topology*, such that:

- $\varnothing, X \in \mathcal{T}$,
- for any collection $\{U_i\}_{i\in I} \subseteq \mathcal{T}$ we have that $\bigcup_{i\in I} U_i \in \mathcal{T}$, and
- for any finite collection $\{U_i\}_{i=1}^n$ we have that $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We will refer to the elements of $\mathcal{T}$ as *open sets*. It is not hard to see that topological spaces exist.

**Example 1.1.**

- If $X$ is any metric space, then one can place a topology on $X$ by letting
$$\mathcal{T} = \{B_\varepsilon(x) : x \in X, \varepsilon > 0\}.$$
In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are topological spaces under the usual metric.
- If $Y$ is a subset of any topological space $X$, then $Y$ inherits a subspace topology, where the open sets in $Y$ are just the open sets in $X$ which intersect $Y$. In particular, the *n-sphere*
$$S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\} \subseteq \mathbb{R}^{n+1}$$
is a topological space.
- If $X$ is any set, then there are always two ways to topologize $X$: the *discrete topology* has open sets every possible subset of $X$, and the *trivial topology* has open sets just $\varnothing$ and $X$.

Well, since topological spaces exist, and we can explicitly construct them, let's move on to the second point. Can we *classify* topological spaces?

There is a strict way to attempt to approach this question: classify spaces up to *homeomorphism*. Recall that a *continuous function* $f : X \to Y$ is a function such that the preimage $f^{-1}(U) \subseteq X$ of any open set $U \subseteq Y$ is open, and a *homeomorphism* is a continuous function such that there exists another continuous function $g : Y \to X$, which we'll call the *inverse*, such that

$$g \circ f = \mathrm{id}_X : X \to X, \quad f \circ g = \mathrm{id}_Y : Y \to Y.$$

⚠**Warning** ⚠ **1.2.** It is not enough for a continuous function to be a bijection to be a homeomorphism.

**Exercise 1.3.** Construct a continuous bijection which is not a homeomorphism.

It is not an easy task to determine whether or not two random spaces are homeomorphic. If we want to be really general and consider all topological spaces (or at least most of them), then classifying them *up to homeomorphism* seems a little out of reach.

There are two clear directions to move forward from this obstacle. On the one hand, we can put more structure on our spaces and impose a stricter notion of equivalence. For example, we could only consider *manifolds*, which are topological spaces that are locally modeled by $\mathbb{R}^n$, or we could consider *differentiable manifolds*, which have structures amenable to the tools of calculus, or we could consider *varieties*, which are the zero-loci of systems of polynomial equations. Each of these types of spaces has a more fine-tuned notion of equivalence which allows us to better classify that type of space.

This direction will not be the focus of this course. Instead, we will *loosen* our notion of equivalence and study spaces up to *homotopy equivalence*. If you didn't think that topology was flexible enough, what with the open interval being homeomorphic to the real number line, well then get ready for some real gymnastics.

Let $f, g : X \to Y$ be continuous maps. A *homotopy* between $f$ and $g$ is a continous function

$$H : X \times I \to Y$$

where $I = [0,1]$ denotes the unit interval (which we think of as a time parameter), such that at time $t = 0$, we have $H(x,0) = f(x)$, and at time $t = 1$, we have $H(x,1) = g(x)$. We will often use the notation $f \simeq g$ if there is a homotopy from $f$ to $g$ and say that they are *homotopic*.

**Example 1.4.** Define two continuous maps $f, g : \mathbb{R} \to \mathbb{R}^2$ by

$$f(x) = (x, x^2), \quad g(x) = (x, x).$$

We can define a homotopy between $f$ and $g$ quite easily: let $H : \mathbb{R} \times I \to \mathbb{R}^2$ be defined by

$$H(x,t) = (x, x^2 - tx^2 + tx).$$

Then $H(x,0) = (x,x^2) = f(x)$ and $H(x,1) = (x,x) = g(x)$. This can be seen graphically here `https://www.desmos.com/calculator/acoupu1hy1`.

**Exercise 1.5.** Show that the notion of homotopy defines an equivalence relation on the set of continuous functions from $X$ to $Y$ by showing the following:

- If $f : X \to Y$ is any function, then there is a homotopy $H : X \times I \to Y$ from $f$ to itself.
- If $H : X \times I \to Y$ is a homotopy from $f$ to $g$, then there is a homotopy $H' : X \times I \to Y$ from $g$ to $f$.
- If $f, g, h : X \to Y$ are continuous functions such that $f \simeq g$ and $g \simeq h$, then there is a homotopy $H : X \times I \to Y$ from $f$ to $h$.

A *homotopy equivalence* is a continous function $f : X \to Y$ such that there exists another continuous function $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X : X \to X, \quad f \circ g \simeq \mathrm{id}_Y : Y \to Y.$$

We will often write that $X \simeq Y$ if $X$ is homotopy equivalent to $Y$.

**Example 1.6.** Consider the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ from the previous example. Essentially the same homotopy between the functions $f$ and $g$ shows that the subspaces

$$\mathrm{im}(f), \mathrm{im}(g) \subseteq \mathbb{R}^2,$$

i.e. the graphs of the functions

Notice that this is very similar to the notion of homeomorphism; we have just required that the compositions be *homotopic* to the identity instead of equal. And indeed, this is a weaker notion.

**Proposition 1.7.** *Let $f : X \to Y$ be a homeomorphism. Then $f$ is a homotopy equivalence.*

*Proof.* Since $f$ is a homeomorphism, there is a continuous function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Define a function $H : X \times I \to X$ by $H(x,t) = (g \circ f)(x)$. This is continuous: if $U \subseteq X$ is open, then

$$H^{-1}(U) = (g \circ f)^{-1}(U) \times I = (\mathrm{id}_X)^{-1}(U) \times I = U \times I.$$

This is open in the product topology on $X \times I$. Moreover, $H(x,0) = (g \circ f)(x)$, and $H(x,1) = (g \circ f)(x) = \mathrm{id}_x(x) = x$, so $H$ is a homotopy between $g \circ f$ and $\mathrm{id}_X$. The same argument works to show that $f \circ g \simeq \mathrm{id}_Y$. □

To illustrate how loose of a notion homotopy equivalence is, consider the following example.

**Example 1.8.** Let $\overline{\mathbb{D}^n}$ be the closed unit disc in $\mathbb{R}^n$, regardless of $n$, and let $*$ denote, well, a point. It is not hard to see that $\overline{\mathbb{D}^n}$ is homotopy equivalent to $*$: let $f : \overline{\mathbb{D}^n} \to *$ be the only map possible (which sends every point in the disc to the point) and let $g : * \to \overline{\mathbb{D}^n}$ send $* \mapsto 0$ (or honestly to any point in the disc). Then we have

$$f \circ g : * \to *, \quad (f \circ g)(*) = *,$$

and

$$g \circ f : \overline{\mathbb{D}^n} \to \overline{\mathbb{D}^n}, \quad (g \circ f)(x) = g(*) = 0.$$

It is worth checking (and not hard) that

$$H : * \times I \to *, \quad H(*,t) = *,$$

and

$$H' : \overline{\mathbb{D}^n} \times I \to \overline{\mathbb{D}^n}, \quad H(x,t) = t \cdot x$$

give the desired homotopy equivalences $g \circ f \simeq \mathrm{id}_{\overline{\mathbb{D}^n}}$ and $f \circ g \simeq \mathrm{id}_*$.

Something the above example shows is: if you had any preconceived notion of dimension, just know that homotopy equivalences couldn't care less! We will call a space which is homotopy equivalent to a point *contractible*. Notice that the above argument also shows that $\mathbb{R}^n$ itself is contractible.

⚠**Warning** ⚠ **1.9.** It is NOT TRUE that if $X$ is contractible and $Y \subseteq X$ is a subspace, then $Y$ is contractible.

It turns out that, except in some particularly cherry-picked examples like the one above, it is still not very easy to determine if two randomly selected spaces are homotopy equivalent. There are a lot of ways to try and address this problem; one of the ways we will address it is by introducing *invariants* known as *homotopy groups*.

**Lecture 2.**

We will explore this more thoroughly throughout the course; for now, we will say that the $n^{th}$ homotopy group of a space, denoted $\pi_n(X)$, is a calculable invariant which roughly measures the number of $n$-dimensional holes in $X$. The key point is that instead of telling us when two spaces are homotopy equivalent, these groups are a nice way to tell when two spaces are NOT homotopy equivalent!

**Theorem 1.10.** *Suppose that for some $n > 0$, $\pi_n(X) \neq \pi_n(Y)$. Then $X$ is not homotopy equivalent to $Y$.*

So, if we can develop tools to compute homotopy groups, then we can try to differentiate spaces. Again, if this were easy, then there would be no field of homotopy theory. But it is in the surprising complexity of this problem that some really beautiful mathematics has been discovered!

Let's look at our favorite (or at least my favorite) spaces: the spheres $S^n$. These spaces are very simple to define, and for the most part we can even imagine what they look like (or at least pretend). However, the homotopy groups of spheres have a well earned reputation for being chaotic, full of mystery, and are wildly unpredictable. Table 1 has a few examples.

| $\pi_i(S^n)$ | $S^1$ | $S^2$ | $S^3$ | $S^4$ | $S^5$ | $S^6$ | $S^7$ |
|---|---|---|---|---|---|---|---|
| $\pi_1(-)$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\pi_2(-)$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 |
| $\pi_3(-)$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 |
| $\pi_4(-)$ | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| $\pi_5(-)$ | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}$ | 0 | 0 |
| $\pi_6(-)$ | 0 | $\mathbb{Z}/12$ | $\mathbb{Z}/12$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}$ | 0 |
| $\pi_7(-)$ | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z} \times \mathbb{Z}/12$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}$ |
| $\pi_8(-)$ | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2 \times \mathbb{Z}/2$ | $\mathbb{Z}/24$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ |
| $\pi_9(-)$ | 0 | $\mathbb{Z}/3$ | $\mathbb{Z}/3$ | $\mathbb{Z}/2 \times \mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/24$ | $\mathbb{Z}/2$ |
| $\pi_{10}(-)$ | 0 | $\mathbb{Z}/15$ | $\mathbb{Z}/15$ | $\mathbb{Z}/24 \times \mathbb{Z}/3$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/24$ |

TABLE 1. Some homotopy groups of spheres. What could it all mean?

We'll talk later about what these particular values mean, but maybe you can already start to see some patterns. Here are a few observations that I'll point out:

- The only non-trivial homotopy group of the circle $S^1$ is $\pi_1(S^1) = \mathbb{Z}$. Topologically this is like saying that there are no "higher dimensional hole" in $S^1$.
- If $n < k$, then $\pi_n(S^k) = 0$. Topologically, this is like saying there smallest hole in $S^k$ is of dimension $k$.
- At least in the values depicted here, we see that $\pi_n(S^n) = \mathbb{Z}$.

- For $k > 1$, we do not see that $\pi_n(S^k) = 0$ for $n > k$, as opposed to the case for $S^1$. So, this is saying that there are "high-dimensional holes" in $S^k$.
- The values given are *almost* all finite.

We will try our best to understand and more rigorously study some of these concepts throughout this course.

Another feature of homotopy theory, which is another core tenant of this course, is that there are operations on spaces themselves that give us relations between homotopy groups and can inform us about homotopy equivalences. If homotopy groups are an assignment of algebra to a topological problem, then space-level operations are an *extraction* of algebra from within a topological problem. To avoid getting too side-winded, let me just give a brief example.

**Example 1.11.** Consider the torus $\mathbb{T}$ and the 2-sphere $S^2$. Both of these objects live in $\mathbb{R}^3$ and seem to have a "hole". How can we tell if they are homotopy equivalent or not without explicitly trial-and-error-ing our way through this problem?

Another way to write the torus is as $\mathbb{T} = S^1 \times S^1$; that is, we can decompose the torus as a product of two better understood spaces (at least, if we accept the above table). This is very nice: homotopy groups *respect products*. That is $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$. In our case, this tells us that

$$\pi_1(\mathbb{T}) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}.$$

But our above table says that $\pi_1(S^2) = 0$, so there can be no homotopy equivalence between the torus and the 2-sphere.

One may organize this course into 3 different themes:

- Attaching algebraic invariants to topological spaces;
- Studying algebraic phenomena exhibited by topological spaces;
- Understanding the passage from topology to algebra and vice versa.

This last point really means: we will develop and use the tools of category theory to study the homotopy theory of topological spaces.

## 2. The Fundamental Group

**Lecture 3.**

Let $X$ be a topological space. A *path* in $X$ is a continuous function $\gamma : I \to X$, where $I \subseteq \mathbb{R}$ is the closed unit interval. Two paths $\gamma, \varphi : I \to X$ are *composable* if $\gamma(1) = \varphi(0)$, i.e. if one ends at the starting point of the other. Define their *product* as the path $\varphi \cdot \gamma : I \to X$ given by

$$\varphi \cdot \gamma(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \varphi(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

Notice that we should read function product from right to left, just as we read function composition. Not everyone follows this convention, so beware!

Paths give us a way to "feel out" the structure of a space. We only want to work with paths up to homotopy in this class, as we want to work with everything up to homotopy, but we encounter a bit of a snag.

**Exercise 2.1.** Let $X$ be path connected. If $\gamma : I \to X$ and $\varphi : I \to X$ are any paths, then there is a homotopy $H : I \times I \to X$ from $\gamma$ to $\varphi$. (**Hint:** If $X$ is path-connected, then there is a path from $\gamma(0)$ to $\varphi(0)$. Use this path to go from $\gamma$ to $\varphi$.)

This implies that, so as not to make everything trivial in a path-connected space, we should make a slight more refined notion of homotopy for paths. A *path homotopy* between two paths $\gamma, \varphi : I \to X$ is a homotopy $H : I \times I \to X$ from $\gamma$ to $\varphi$ such that

$$H(0, t) = \gamma(0) = \varphi(0), \quad H(1, t) = \gamma(1) = \varphi(1)$$

In other words, the ends of the interval $I$ are sent to the same points in $X$ by both $\gamma$ and $\varphi$, and for any time $t_0$, the path $f_{t_0} : I \to X$ given by $f(s) = F(t_0, s)$ also sends the ends of $I$ to the same points as $\gamma$ and $\varphi$ do. Thus a path homotopy only exists for particular paths which share endpoints.

Just as homotopy equivalence defines an equivalence relation, so too does path homotopy equivalence. We will let $[f]$ denote the *path homotopy class* of $f$; these are the paths into $X$ which are path-homotopic to $f$.

A path is called a *loop* if we have that $\gamma(0) = \gamma(1)$. Note that if $\gamma : I \to X$ is a loop, then since we can view $S^1$ as the quotient space $I/0 \sim 1$, we can factor $\gamma$ uniquely through the circle:

$$\begin{array}{ccc} I & \xrightarrow{\gamma} & X \\ \downarrow & \nearrow_{\overline{\gamma}} & \\ S^1 & & \end{array}$$

The universal property of the quotient implies that $\overline{\gamma}$ is continuous since $\gamma$ is. In fact, this factorization characterizes loops! A path is a loop if and only if the above diagram commutes.

We come to our first invariant. Let $(X, x_0)$ be a based space, which is just a topological space $X$ with a chosen basepoint $x_0$. Note that a morphism of based spaces is just a continuous morphism which sends basepoint to basepoint. The *fundamental group* of $X$ based at $x_0$, denoted $\pi_1(X, x_0)$, is the set of path classes of loops in $X$ based at $x_0$. In other words,

$$\pi_1(X, x_0) = \{[f] | f : (S^1, 1) \to (X, x_0) \text{ based map}\}.$$

There is a constant path $c : S^1 \to X$ where $c(t) = x_0$ for all $t \in S^1$. In general, though, it takes some work to determine what the actual loops are in a space.

**Exercise 2.2.** Show that re-parameterizing paths doesn't actually do anything to the fundamental group. That is, let $\gamma : S^1 \to X$ be some loop. We can lift this along the quotient map $I \to S^1$ to a map $\tilde{\gamma} : I \to X$ where $\tilde{\gamma}(0) = \tilde{\gamma}(1)$. Let $\varphi : I \to I$ be any continuous map where $\varphi(0) = 0$ and $\varphi(1) = 1$. Show that there is a path homotopy from $\tilde{\gamma}$ to $\tilde{\gamma} \circ \varphi$.

The naming of $\pi_1(X, x_0)$ is not accidental. For a path $\gamma : I \to X$, let $\gamma^{\text{rev}} : I \to X$ be the path which "does $\gamma$ in reverse". That is, $\gamma^{\text{rev}}(t) = \gamma(1 - t)$.

**Theorem 2.3.** *Let $\gamma : p \mapsto q$ be a path in a space $X$, and let $\varphi, \psi : I \to X$ be any paths.*

- $[c_p] \cdot [\gamma] = [\gamma] \cdot [c_q] = [\gamma]$
- $[\gamma][\gamma^{rev}] = [c_p]$ *and* $[\gamma^{rev}][\gamma] = [c_q]$
- *Whenever defined, we have* $[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h]$.

**Corollary 2.4.** *The set $\pi_1(X, x_0)$ forms a group under path composition, with identity given by $[c_{x_0}]$.*

⚠**Warning** ⚠ **2.5.** The fundamental group need not be abelian! That is, there are based spaces $(X, x_0)$ and loops $[\gamma], [\varphi] \in \pi_1(X, x_0)$ such that $[\varphi] \cdot [\gamma] \neq [\gamma] \cdot [\varphi]$.

The choice of basepoint is very important here! For example, take $X = Y \amalg \mathbb{R}$, where $Y$ is any space such that $\pi_1(Y, y_0) \neq 0$ for some point $y_0 \in Y$. Then

$$\pi_1(X, y_0) \cong \pi_1(Y, y_0) \neq 0,$$

since any loop in $X$ which is based at a point in $Y$ must be entirely contained in $Y$ as the image of a connected set is connected. For the same argument, if $t \in \mathbb{R}$ is any point, then any loop $S^1 \to X$ based at $t$ must have image entirely contained in $\mathbb{R}$. But we have already seen that $\mathbb{R} \simeq *$, which implies that $\pi_1(\mathbb{R}, t) = 0$. Thus, we also have that $\pi_1(X, t) = 0$.

**Lecture 4.**

However, things are much nicer in path-connected spaces.

**Proposition 2.6.** *Suppose that $X$ is path-connected, $p, q \in X$ are any points, and $\gamma : p \mapsto q$ is a path. Define a function*

$$\Phi : \pi_1(X, p) \to \pi_1(X, q), \quad \Phi([\varphi]) = [\gamma] \cdot [\varphi] \cdot [\gamma^{\text{rev}}].$$

*This is an isomorphism of groups.*

*Proof.* First, we must check that this is a group homomorphism. Observe:

$$\begin{aligned}\Phi([f]\cdot[g]) &= [\gamma]\cdot[f]\cdot[g]\cdot[\gamma^{\text{rev}}]\\ &= [\gamma]\cdot[f]\cdot[\gamma^{\text{rev}}]\cdot[\gamma]\cdot[g]\cdot[\gamma^{\text{rev}}]\\ &= \Phi([f])\cdot\Phi([g]).\end{aligned}$$

For injectivity, suppose that $\Phi([f]) = 0 = [c_q]$. Expanding out our function, this tells us that

$$\Phi([f]) = [\gamma]\cdot[f]\cdot[\gamma^{\text{rev}}] = [c_q].$$

Multiplying on the left by $[\gamma^{\text{rev}}]$ and on the right by $[\gamma]$, we obtain

$$[\gamma^{\text{rev}}]\cdot[\gamma]\cdot[f]\cdot[\gamma^{\text{rev}}]\cdot[\gamma] = [\gamma^{\text{rev}}]\cdot[c_q]\cdot[\gamma].$$

Since $[\gamma]\cdot[\gamma^{\text{rev}}] = [c_q]$, and since $[c_q]$ is the identity of $\pi_1(X,q)$, we can rewrite our equation as

$$[f] = [c_p],$$

giving injectivity.

I will leave surjectivity as an exercise. □

If $(X, x_0)$ is path-connected, we will simply refer to its fundamental group by $\pi_1(X)$, as the above proposition proves that this is independent of basepoint.

The next proposition gives us a way to pass from topology to algebra at the level of functions. Namely, continuous maps of based spaces induce group homomorphisms of fundamental groups.

**Proposition 2.7.** *Let $f : (X, x_0) \to (Y, y_0)$ be a map of based spaces. Then the function*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad f_*([\gamma]) = [f \circ \gamma]$$

*is a group homomorphism. Moreover:*

- *if we have $g : (Y, y_0) \to (Z, z_0)$ continuous, then*
$$(g_* \circ f_*)([\gamma]) = (g \circ f)_*([\gamma])$$
- *if $\mathrm{id}_X : (X, x_0) \to (X, x_0)$ is the identity map, then*
$$(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x_0)}$$

*Proof.* Homework. □

Let's look at some examples!

**Example 2.8.**

- Suppose that $X \simeq *$ is contractible and $x_0 \in X$ is any point. Then there is a homotopy $H : X \times I \to X$ witnessing $\mathrm{id}_X \simeq c_{x_0}$. We can use this homotopy to show that any loop $f : S^1 \to X$ is nullhomotopic. Let $F : S^1 \times I \to X$ be defined by
$$F(r, t) = H(f(r), t).$$
This is continuous, as another way to write this function is as $F = H \circ (f \times \mathrm{id}_I)$. Moreover,
$$F(r, 0) = H(f(r), 0) = \mathrm{id}_X(f(r) = f(r),$$
and
$$F(r, 1) = H(f(r), 1) = c_{x_0}(f(r)) = x_0.$$
Thus, if $X$ is any contractible space, then $\pi_1(X, x_0) = 0$ for any basepoint $X$.
- More generally, and this is slightly more work to show, if $f : X \to Y$ is a homotopy equivalence, then the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism of groups. Thus, homotopy equivalent spaces have the same fundamental groups.

Next, we will prove our first big theorem, and make our first nontrivial fundamental group computation.

**Theorem 2.9.** *The fundamental group of the circle is*

$$\pi_1(S^1) \cong \mathbb{Z}$$

This will require a bit of work. First, define a path $\omega_n : I \to S^1$ by

$$\omega_n(t) = e^{2\pi i \cdot nt},$$

where $n \in \mathbb{Z}$ and we are viewing $S^1 \subseteq \mathbb{C}$. This path is a loop, since $e^0 = e^{2\pi 1 \cdot n} = 1$ for all $n \in \mathbb{Z}$. Geometrically, this is the loop which winds around the circle $n$-times.

**Lecture 5.**

It is straightforward to see that $[\omega_n] \cdot [\omega_m] = [\omega_{n+m}]$. This simple observation is really useful: it implies that the function

$$\phi : \mathbb{Z} \to \pi_1(S^1, 1), \quad \phi(n) = [\omega_n]$$

is a group homomorphism. Our goal now is to show that $\phi$ is an isomorphism.

The intuition here is that the collection of maps $\{\omega_n\}_{n \in \mathbb{Z}}$ acts a lot like the integers, where the number $n \in \mathbb{Z}$ corresponds to going around the circle $n$-times counter-clockwise. The point of our calculation is that that is all that can happen.

- Showing the map $\phi$ is surjective says that every loop in $S^1$ looks like "going around the circle" some number of times.
- Showing the map $\varphi$ is injective says that going around the loop $n$-times and going around the loop $m$-times is never somehow equivalent in $\pi_1(S^1, 1)$ when $n \neq m$.

The "reason" why $\phi$ is an isomorphism is that, locally, $S^1$ just looks like the real numbers $\mathbb{R}$. Since $\mathbb{R} \simeq *$, meaning $\pi_1(\mathbb{R}, 1) = 0$, there can be no weird local behavior of $S^1$ that obfuscates our argument.

There is a continuous map $p : \mathbb{R} \to S^1$ which will feature very heavily in these arguments. This map is defined by

$$p(t) = e^{2\pi i \cdot t},$$

which we can geometrically think of as "winding the real numbers around $S^1$". Consider again the map $\omega_n : I \to S^1$. We can lift this map along $p$ to a map $\tilde{\omega}_n : I \to \mathbb{R}$ where

$$\tilde{\omega}_n(t) = n \cdot t.$$

By "lift along $p$", I mean that the maps $\omega_n$ and $p$ fit into a diagram of the form:

$$\begin{array}{ccc} & & \mathbb{R} \\ & & \big\downarrow p \\ I & \xrightarrow{\omega_n} & S^1 \end{array}$$

and the map $\tilde{\omega}_n$ fills in the triangle and makes the diagram commute:

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\tilde{\omega}_n}{\nearrow} & \big\downarrow p \\ I & \xrightarrow{\omega_n} & S^1 \end{array}$$

In fact, the existence of a lift is not unique to the paths $\gamma_n$. All paths in $S^1$ lift to $\mathbb{R}$, and they lift *uniquely* as soon as we specify any value.

**Proposition 2.10** (Path lifting)**.** *Let $f : I \to S^1$ be any path with $\gamma(0) = x \in S^1$. Let $\tilde{x} \in p^{-1}(x)$ be any point in the fiber over $x$. Then there is a unique lift $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}(0) = \tilde{x}$ and $f = p \circ \tilde{f}$.*

**Exercise 2.11.** Prove the above proposition. Notice that as there are infinitely many points in the fiber $p^{-1}(x) \subseteq \mathbb{R}$ (in fact if $t \in p^{-1}(x)$ then $t + n \in p^{-1}(x)$ for any $n \in \mathbb{Z}$), the choice of lift $\tilde{f}$ is certainly not unique unless we specify $\tilde{f}(0)$.

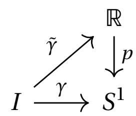
Path lifting is nice. Even stronger is that we can lift entire homotopies from $S^1$ to $\mathbb{R}$.

**Proposition 2.12** (Homotopy path lifting)**.** *Let $H : I \times I \to S^1$ be a path homotopy from $H(s,0)$ and $H(s,1)$, and let $x = H(0,t)$ for any $t \in I$. Let $\tilde{x} \in p^{-1}(x)$. Then there is a unique lift $\tilde{H} : I \times I \to \mathbb{R}$ such that $H = p \circ \tilde{H}$ and $\tilde{H}(0,t) = \tilde{x}$ for all $t \in I$.*

**Remark 2.13.** We will not prove the above proposition. For reference, see section 1.1 of Hatcher's book.

Using these propositions, we will prove that $\pi_1(S^1,1) \cong \mathbb{Z}$.

First, we show that $\phi$ is surjective. Let $\gamma : (I,0) \to (S^1,1)$ be a loop. Path lifting implies that there is a unique lift $\tilde{\gamma} : (I,1) \to (\mathbb{R},0)$ such that $\tilde{\gamma}(0) = 0$.




By commutativity of the above diagram, we have that $p \circ \tilde{\gamma}(1) = \gamma(1) = \gamma(0) = 1$, thus $\tilde{\gamma}(1) \in p^{-1}(1)$. Since $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$, this implies that $\tilde{\gamma}(1)$ is some integer $n$.

There is another path in $\mathbb{R}$ from 0 to $n$: it is the lift $\tilde{\omega}_n : I \to \mathbb{R}$. We have a homotopy from $\tilde{\gamma}$ to $\tilde{\omega}_n$:

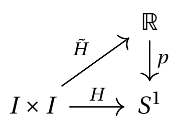
$$H : I \times I \to \mathbb{R}, \quad H(s,t) = (1-t)\tilde{\gamma}(s) + t \cdot \tilde{\omega}_n(s).$$

Now we can just compose with $p$, getting a map $p \circ H : I \times I \to \mathbb{R} \to S^1$. By construction, this map exhibits a homotopy between $\gamma$ and $\omega_n$. Hence $[\omega_n] = [\gamma]$, so $\phi$ is surjective.

Now, suppose that $\gamma : I \to S^1$ is some path such that $\gamma \simeq \omega_n$ and $\gamma \simeq \omega_m$. Homotopy equivalence is transitive, so there is a homotopy

$$H : I \times I \to S^1$$

from $\omega_n$ to $\omega_m$. By Homotopy path lifting, there is a unique lift $\tilde{H} : I \times I \to \mathbb{R}$ such that $\tilde{H}(0,t) = 0$.




By uniqueness of path lifting, since $\tilde{H}(s,0)$ is a lift of $\omega_n$ which starts at 0 and ends at $n$, we must have that $\tilde{H}(s,0) = \tilde{\omega}_n(s)$. Similarly, $\tilde{H}(s,1) = \tilde{\omega}_m(s)$. Since $\tilde{H}$ is a path homotopy, we must have that $\tilde{H}(1,t)$ is a fixed point in $\mathbb{R}$. When $t = 0$, we see that $\tilde{H}(1,0) = \tilde{\omega}_n(1) = n$. When $t = 1$, we see that $\tilde{H}(1,1) = \tilde{\omega}_m(1) = m$. Therefore, we must have that $n = m$, hence $\phi$ is injective, finishing the proof.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON
*Email address*: jackmann@uw.edu