MATH 480: HOMOTOPY THEORY

JACKSON MORRIS

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1. MOTIVATION/BACKGROUND

Lecture 1.

The things that mathematicians care about can often be distinguished by two qualities: *existence* and *uniqueness*. That is, suppose a mathematician thinks of some concept X. There are two obvious questions to ask:

- Does such a concept X actually exist? If so, can we explicitly construct it?
- If a concept X exists, is it unique? If it isn't unique, can we classify all such types of concepts X?

In this course, we will be studying topological spaces. Recall that a *topological space* is a set X together with a collection $\mathcal{T}$ of subsets, called the *topology*, such that:

- $\varnothing, X \in \mathcal{T}$,
- for any collection $\{U_i\}_{i\in I} \subseteq \mathcal{T}$ we have that $\bigcup_{i\in I} U_i \in \mathcal{T}$, and
- for any finite collection $\{U_i\}_{i=1}^n$ we have that $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We will refer to the elements of $\mathcal{T}$ as *open sets*. It is not hard to see that topological spaces exist.

Example 1.1.

- If X is any metric space, then one can place a topology on X by letting
$$\mathcal{T} = \{B_\varepsilon(x) : x \in X, \varepsilon > 0\}.$$
In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are topological spaces under the usual metric.
- If Y is a subset of any topological space X, then Y inherits a subspace topology, where the open sets in Y are just the open sets in X which intersect Y. In particular, the *n-sphere*
$$S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\} \subseteq \mathbb{R}^{n+1}$$
is a topological space.
- If X is any set, then there are always two ways to topologize X: the *discrete topology* has open sets every possible subset of X, and the *trivial topology* has open sets just $\varnothing$ and X.

Well, since topological spaces exist, and we can explicitly construct them, let's move on to the second point. Can we *classify* topological spaces?

There is a strict way to attempt to approach this question: classify spaces up to *homeomorphism*. Recall that a *continuous function* $f : X \to Y$ is a function such that the preimage $f^{-1}(U) \subseteq X$ of any open set $U \subseteq Y$ is open, and a *homeomorphism* is a continuous function such that there exists another continuous function $g : Y \to X$, which we'll call the *inverse*, such that

$$g \circ f = \mathrm{id}_X : X \to X, \quad f \circ g = \mathrm{id}_Y : Y \to Y.$$

⚠Warning ⚠ 1.2. It is not enough for a continuous function to be a bijection to be a homeomorphism.

Exercise 1.3. Construct a continuous bijection which is not a homeomorphism.

It is not an easy task to determine whether or not two random spaces are homeomorphic. If we want to be really general and consider all topological spaces (or at least most of them), then classifying them *up to homeomorphism* seems a little out of reach.

There are two clear directions to move forward from this obstacle. On the one hand, we can put more structure on our spaces and impose a stricter notion of equivalence. For example, we could only consider *manifolds*, which are topological spaces that are locally modeled by $\mathbb{R}^n$, or we could consider *differentiable manifolds*, which have structures amenable to the tools of calculus, or we could consider *varieties*, which are the zero-loci of systems of polynomial equations. Each of these types of spaces has a more fine-tuned notion of equivalence which allows us to better classify that type of space.

This direction will not be the focus of this course. Instead, we will *loosen* our notion of equivalence and study spaces up to *homotopy equivalence*. If you didn't think that topology was flexible enough, what with the open interval being homeomorphic to the real number line, well then get ready for some real gymnastics.

Let $f,g: X \to Y$ be continuous maps. A *homotopy* between f and g is a continous function

$$H: X \times I \to Y$$

where $I = [0,1]$ denotes the unit interval (which we think of as a time parameter), such that at time $t = 0$, we have $H(x,0) = f(x)$, and at time $t = 1$, we have $H(x,1) = g(x)$. We will often use the notation $f \simeq g$ if there is a homotopy from f to g and say that they are *homotopic*.

Example 1.4. Define two continuous maps $f,g: \mathbb{R} \to \mathbb{R}^2$ by

$$f(x) = (x, x^2), \quad g(x) = (x,x).$$

We can define a homotopy between f and g quite easily: let $H: \mathbb{R} \times I \to \mathbb{R}^2$ be defined by

$$H(x,t) = (x, x^2 - tx^2 + tx).$$

Then $H(x,0) = (x,x^2) = f(x)$ and $H(x,1) = (x,x) = g(x)$. This can be seen graphically here `https://www.desmos.com/calculator/acoupu1hy1`.

Exercise 1.5. Show that the notion of homotopy defines an equivalence relation on the set of continuous functions from X to Y by showing the following:

- If $f: X \to Y$ is any function, then there is a homotopy $H: X \times I \to Y$ from f to itself.
- If $H: X \times I \to Y$ is a homotopy from f to g, then there is a homotopy $H': X \times I \to Y$ from g to f.
- If $f,g,h: X \to Y$ are continuous functions such that $f \simeq g$ and $g \simeq h$, then there is a homotopy $H: X \times I \to Y$ from f to h.

A *homotopy equivalence* is a continous function $f: X \to Y$ such that there exists another continuous function $g: Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X : X \to X, \quad f \circ g \simeq \mathrm{id}_Y : Y \to Y.$$

We will often write that $X \simeq Y$ if X is homotopy equivalent to Y.

Example 1.6. Consider the functions $f,g: \mathbb{R} \to \mathbb{R}^2$ from the previous example. Essentially the same homotopy between the functions f and g shows that the subspaces

$$\mathrm{im}(f), \mathrm{im}(g) \subseteq \mathbb{R}^2,$$

i.e. the graphs of the functions

Notice that this is very similar to the notion of homeomorphism; we have just required that the compositions be *homotopic* to the identity instead of equal. And indeed, this is a weaker notion.

Proposition 1.7. *Let $f: X \to Y$ be a homeomorphism. Then f is a homotopy equivalence.*

Proof. Since f is a homeomorphism, there is a continuous function $g: Y \to X$ such that $g \circ f = \mathrm{id}_{\mathrm{X}}$ and $f \circ g = \mathrm{id}_Y$. Define a function $H: X \times I \to X$ by $H(x,t) = (g \circ f)(x)$. This is continuous: if $U \subseteq X$ is open, then

$$H^{-1}(U) = (g \circ f)^{-1}(U) \times I = (\mathrm{id}_X)^{-1}(U) \times I = U \times I.$$

This is open in the product topology on $X \times I$. Moreover, $H(x,0) = (g \circ f)(x)$, and $H(x,1) = (g \circ f)(x) = \mathrm{id}_x(x) = x$, so H is a homotopy between $g \circ f$ and id_X. The same argument works to show that $f \circ g \simeq \mathrm{id}_Y$. □

To illustrate how loose of a notion homotopy equivalence is, consider the following example.

Example 1.8. Let $\overline{\mathbb{D}^n}$ be the closed unit disc in $\mathbb{R}^n$, regardless of n, and let $*$ denote, well, a point. It is not hard to see that $\overline{\mathbb{D}^n}$ is homotopy equivalent to $*$: let $f: \overline{\mathbb{D}^n} \to *$ be the only map possible (which sends every point in the disc to the point) and let $g: * \to \overline{\mathbb{D}^n}$ send $* \mapsto 0$ (or honestly to any point in the disc). Then we have

$$f \circ g : * \to *, \quad (f \circ g)(*) = *,$$

and

$$g \circ f : \overline{\mathbb{D}^n} \to \overline{\mathbb{D}^n}, \quad (g \circ f)(x) = g(*) = 0.$$

It is worth checking (and not hard) that

$$H: * \times I \to *, \quad H(*,t) = *,$$

and

$$H' : \overline{\mathbb{D}^n} \times I \to \overline{\mathbb{D}^n}, \quad H(x,t) = t \cdot x$$

give the desired homotopy equivalences $g \circ f \simeq \mathrm{id}_{\overline{\mathbb{D}^n}}$ and $f \circ g \simeq \mathrm{id}_*$.

Something the above example shows is: if you had any preconceived notion of dimension, just know that homotopy equivalences couldn't care less! We will call a space which is homotopy equivalent to a point *contractible*. Notice that the above argument also shows that $\mathbb{R}^n$ itself is contractible.

⚠**Warning** ⚠ **1.9.** It is NOT TRUE that if X is contractible and $Y \subseteq X$ is a subspace, then Y is contractible.

It turns out that, except in some particularly cherry-picked examples like the one above, it is still not very easy to determine if two randomly selected spaces are homotopy equivalent. There are a lot of ways to try and address this problem; one of the ways we will address it is by introducing *invariants* known as *homotopy groups*.

Lecture 2.

We will explore this more thoroughly throughout the course; for now, we will say that the n^{th} homotopy group of a space, denoted $\pi_n(X)$, is a calculable invariant which roughly measures the number of n-dimensional holes in X. The key point is that instead of telling us when two spaces are homotopy equivalent, these groups are a nice way to tell when two spaces are NOT homotopy equivalent!

Theorem 1.10. *Suppose that for some $n > 0$, $\pi_n(X) \neq \pi_n(Y)$. Then X is not homotopy equivalent to Y.*

So, if we can develop tools to compute homotopy groups, then we can try to differentiate spaces. Again, if this were easy, then there would be no field of homotopy theory. But it is in the surprising complexity of this problem that some really beautiful mathematics has been discovered!

Let's look at our favorite (or at least my favorite) spaces: the spheres S^n. These spaces are very simple to define, and for the most part we can even imagine what they look like (or at least pretend). However, the homotopy groups of spheres have a well earned reputation for being chaotic, full of mystery, and are wildly unpredictable. Table 1 has a few examples.

$\pi_i(S^n)$	S^1	S^2	S^3	S^4	S^5	S^6	S^7
$\pi_1(-)$	$\mathbb{Z}$	0	0	0	0	0	0
$\pi_2(-)$	0	$\mathbb{Z}$	0	0	0	0	0
$\pi_3(-)$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
$\pi_4(-)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0
$\pi_5(-)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0
$\pi_6(-)$	0	$\mathbb{Z}/12$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
$\pi_7(-)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$
$\pi_8(-)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\pi_9(-)$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$
$\pi_{10}(-)$	0	$\mathbb{Z}/15$	$\mathbb{Z}/15$	$\mathbb{Z}/24 \times \mathbb{Z}/3$	$\mathbb{Z}/2$	0	$\mathbb{Z}/24$

TABLE 1. Some homotopy groups of spheres. What could it all mean?

We'll talk later about what these particular values mean, but maybe you can already start to see some patterns. Here are a few observations that I'll point out:

- The only non-trivial homotopy group of the circle S^1 is $\pi_1(S^1) = \mathbb{Z}$. Topologically this is like saying that there are no "higher dimensional hole" in S^1.
- If $n < k$, then $\pi_n(S^k) = 0$. Topologically, this is like saying there smallest hole in S^k is of dimension k.
- At least in the values depicted here, we see that $\pi_n(S^n) = \mathbb{Z}$.

- For $k > 1$, we do not see that $\pi_n(S^k) = 0$ for $n > k$, as opposed to the case for S^1. So, this is saying that there are "high-dimensional holes" in S^k.
- The values given are *almost* all finite.

We will try our best to understand and more rigorously study some of these concepts throughout this course.

Another feature of homotopy theory, which is another core tenant of this course, is that there are operations on spaces themselves that give us relations between homotopy groups and can inform us about homotopy equivalences. If homotopy groups are an assignment of algebra to a topological problem, then space-level operations are an *extraction* of algebra from within a topological problem. To avoid getting too side-winded, let me just give a brief example.

Example 1.11. Consider the torus $\mathbb{T}$ and the 2-sphere S^2. Both of these objects live in $\mathbb{R}^3$ and seem to have a "hole". How can we tell if they are homotopy equivalent or not without explicitly trial-and-error-ing our way through this problem?

Another way to write the torus is as $\mathbb{T} = S^1 \times S^1$; that is, we can decompose the torus as a product of two better understood spaces (at least, if we accept the above table). This is very nice: homotopy groups *respect products*. That is $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$. In our case, this tells us that

$$\pi_1(\mathbb{T}) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}.$$

But our above table says that $\pi_1(S^2) = 0$, so there can be no homotopy equivalence between the torus and the 2-sphere.

One may organize this course into 3 different themes:

- Attaching algebraic invariants to topological spaces;
- Studying algebraic phenomena exhibited by topological spaces;
- Understanding the passage from topology to algebra and vice versa.

This last point really means: we will develop and use the tools of category theory to study the homotopy theory of topological spaces.

2. The fundamental group

Lecture 3.

Let X be a topological space. A *path* in X is a continuous function $\gamma : I \to X$, where $I \subseteq \mathbb{R}$ is the closed unit interval. Two paths $\gamma, \varphi : I \to X$ are *composable* if $\gamma(1) = \varphi(0)$, i.e. if one ends at the starting point of the other. Define their *product* as the path $\varphi \cdot \gamma : I \to X$ given by

$$\varphi \cdot \gamma(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \varphi(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

Notice that we should read function product from right to left, just as we read function composition. Not everyone follows this convention, so beware!

Paths give us a way to "feel out" the structure of a space. We only want to work with paths up to homotopy in this class, as we want to work with everything up to homotopy, but we encounter a bit of a snag.

Exercise 2.1. Let X be path connected. If $\gamma : I \to X$ and $\varphi : I \to X$ are any paths, then there is a homotopy $H : I \times I \to X$ from γ to φ. (**Hint:** If X is path-connected, then there is a path from $\gamma(0)$ to $\varphi(0)$. Use this path to go from γ to φ.)

This implies that, so as not to make everything trivial in a path-connected space, we should make a slight more refined notion of homotopy for paths. A *path homotopy* between two paths $\gamma, \varphi : I \to X$ is a homotopy $H : I \times I \to X$ from γ to φ such that

$$H(0, t) = \gamma(0) = \varphi(0), \quad H(1, t) = \gamma(1) = \varphi(1)$$

In other words, the ends of the interval I are sent to the same points in X by both γ and φ, and for any time t_0, the path $f_{t_0} : I \to X$ given by $f(s) = F(t_0, s)$ also sends the ends of I to the same points as γ and φ do. Thus a path homotopy only exists for particular paths which share endpoints.

Just as homotopy equivalence defines an equivalence relation, so too does path homotopy equivalence. We will let $[f]$ denote the *path homotopy class* of f; these are the paths into X which are path-homotopic to f.

A path is called a *loop* if we have that $\gamma(0) = \gamma(1)$. Note that if $\gamma : I \to X$ is a loop, then since we can view S^1 as the quotient space $I/0 \sim 1$, we can factor γ uniquely through the circle:

$$\begin{array}{ccc} I & \xrightarrow{\gamma} & X \\ \downarrow & \nearrow_{\overline{\gamma}} & \\ S^1 & & \end{array}$$

The universal property of the quotient implies that $\overline{\gamma}$ is continuous since γ is. In fact, this factorization characterizes loops! A path is a loop if and only if the above diagram commutes.

We come to our first invariant. Let (X, x_0) be a based space, which is just a topological space X with a chosen basepoint x_0. Note that a morphism of based spaces is just a continuous morphism which sends basepoint to basepoint. The *fundamental group* of X based at x_0, denoted $\pi_1(X, x_0)$, is the set of path classes of loops in X based at x_0. In other words,

$$\pi_1(X, x_0) = \{[f] | f : (S^1, 1) \to (X, x_0) \text{ based map}\}.$$

There is a constant path $c : S^1 \to X$ where $c(t) = x_0$ for all $t \in S^1$. In general, though, it takes some work to determine what the actual loops are in a space.

Exercise 2.2. Show that re-parameterizing paths doesn't actually do anything to the fundamental group. That is, let $\gamma : S^1 \to X$ be some loop. We can lift this along the quotient map $I \to S^1$ to a map $\tilde{\gamma} : I \to X$ where $\tilde{\gamma}(0) = \tilde{\gamma}(1)$. Let $\varphi : I \to I$ be any continuous map where $\varphi(0) = 0$ and $\varphi(1) = 1$. Show that there is a path homotopy from $\tilde{\gamma}$ to $\tilde{\gamma} \circ \varphi$.

The naming of $\pi_1(X, x_0)$ is not accidental. For a path $\gamma : I \to X$, let $\gamma^{\text{rev}} : I \to X$ be the path which "does γ in reverse". That is, $\gamma^{\text{rev}}(t) = \gamma(1 - t)$.

Theorem 2.3. *Let $\gamma : p \mapsto q$ be a path in a space X, and let $\varphi, \psi : I \to X$ be any paths.*

- $[c_p] \cdot [\gamma] = [\gamma] \cdot [c_q] = [\gamma]$
- $[\gamma][\gamma^{rev}] = [c_p]$ *and* $[\gamma^{rev}][\gamma] = [c_q]$
- *Whenever defined, we have* $[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h]$.

Corollary 2.4. *The set $\pi_1(X, x_0)$ forms a group under path composition, with identity given by $[c_{x_0}]$.*

⚠Warning ⚠ 2.5. The fundamental group need not be abelian! That is, there are based spaces (X, x_0) and loops $[\gamma], [\varphi] \in \pi_1(X, x_0)$ such that $[\varphi] \cdot [\gamma] \neq [\gamma] \cdot [\varphi]$.

The choice of basepoint is very important here! For example, take $X = Y \amalg \mathbb{R}$, where Y is any space such that $\pi_1(Y, y_0) \neq 0$ for some point $y_0 \in Y$. Then

$$\pi_1(X, y_0) \cong \pi_1(Y, y_0) \neq 0,$$

since any loop in X which is based at a point in Y must be entirely contained in Y as the image of a connected set is connected. For the same argument, if $t \in \mathbb{R}$ is any point, then any loop $S^1 \to X$ based at t must have image entirely contained in $\mathbb{R}$. But we have already seen that $\mathbb{R} \simeq *$, which implies that $\pi_1(\mathbb{R}, t) = 0$. Thus, we also have that $\pi_1(X, t) = 0$.

Lecture 4.

However, things are much nicer in path-connected spaces.

Proposition 2.6. *Suppose that X is path-connected, $p, q \in X$ are any points, and $\gamma : p \mapsto q$ is a path. Define a function*

$$\Phi : \pi_1(X, p) \to \pi_1(X, q), \quad \Phi([\varphi]) = [\gamma] \cdot [\varphi] \cdot [\gamma^{\text{rev}}].$$

This is an isomorphism of groups.

Proof. First, we must check that this is a group homomorphism. Observe:

$$\begin{aligned}\Phi([f]\cdot[g]) &= [\gamma]\cdot[f]\cdot[g]\cdot[\gamma^{\text{rev}}]\\ &= [\gamma]\cdot[f]\cdot[\gamma^{\text{rev}}]\cdot[\gamma]\cdot[g]\cdot[\gamma^{\text{rev}}]\\ &= \Phi([f])\cdot\Phi([g]).\end{aligned}$$

For injectivity, suppose that $\Phi([f]) = 0 = [c_q]$. Expanding out our function, this tells us that

$$\Phi([f]) = [\gamma]\cdot[f]\cdot[\gamma^{\text{rev}}] = [c_q].$$

Multiplying on the left by $[\gamma^{\text{rev}}]$ and on the right by $[\gamma]$, we obtain

$$[\gamma^{\text{rev}}]\cdot[\gamma]\cdot[f]\cdot[\gamma^{\text{rev}}]\cdot[\gamma] = [\gamma^{\text{rev}}]\cdot[c_q]\cdot[\gamma].$$

Since $[\gamma]\cdot[\gamma^{\text{rev}}] = [c_q]$, and since $[c_q]$ is the identity of $\pi_1(X,q)$, we can rewrite our equation as

$$[f] = [c_p],$$

giving injectivity.

I will leave surjectivity as an exercise. □

If (X,x_0) is path-connected, we will simply refer to its fundamental group by $\pi_1(X)$, as the above proposition proves that this is independent of basepoint.

The next proposition gives us a way to pass from topology to algebra at the level of functions. Namely, continuous maps of based spaces induce group homomorphisms of fundamental groups.

Proposition 2.7. *Let $f:(X,x_0)\to(Y,y_0)$ be a map of based spaces. Then the function*

$$f_*:\pi_1(X,x_0)\to\pi_1(Y,y_0),\quad f_*([\gamma]) = [f\circ\gamma]$$

is a group homomorphism. Moreover:

- *if we have $g:(Y,y_0)\to(Z,z_0)$ continuous, then*

$$(g_*\circ f_*)([\gamma]) = (g\circ f)_*([\gamma])$$

- *if* $\mathrm{id}_X:(X,x_0)\to(X,x_0)$ *is the identity map, then*

$$(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x_0)}$$

Proof. Homework. □

Let's look at some examples!

Example 2.8.

- Suppose that $X\simeq *$ is contractible and $x_0\in X$ is any point. Then there is a homotopy $H: X\times I\to X$ witnessing $\mathrm{id}_X\simeq c_{x_0}$. We can use this homotopy to show that any loop $f:S^1\to X$ is nullhomotopic. Let $F:S^1\times I\to X$ be defined by

$$F(r,t) = H(f(r),t).$$

This is continuous, as another way to write this function is as $F = H\circ(f\times\mathrm{id}_I)$. Moreover,

$$F(r,0) = H(f(r),0) = \mathrm{id}_X(f(r) = f(r),$$

and

$$F(r,1) = H(f(r),1) = c_{x_0}(f(r)) = x_0.$$

Thus, if X is any contractible space, then $\pi_1(X,x_0) = 0$ for any basepoint X.
- More generally, and this is slightly more work to show, if $f: X\to Y$ is a homotopy equivalence, then the induced map $f_*:\pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ is an isomorphism of groups. Thus, homotopy equivalent spaces have the same fundamental groups.

2.1. **The fundamental group of the circle.** Next, we will prove our first big theorem, and make our first nontrivial fundamental group computation.

Theorem 2.9. *The fundamental group of the circle is*

$$\pi_1(S^1) \cong \mathbb{Z}$$

This will require a bit of work. First, define a path $\omega_n : I \to S^1$ by

$$\omega_n(t) = e^{2\pi i\cdot nt},$$

where $n \in \mathbb{Z}$ and we are viewing $S^1 \subseteq \mathbb{C}$. This path is a loop, since $e^0 = e^{2\pi 1\cdot n} = 1$ for all $n \in \mathbb{Z}$. Geometrically, this is the loop which winds around the circle n-times.

Lecture 5.

It is straightforward to see that $[\omega_n]\cdot[\omega_m] = [\omega_{n+m}]$. This simple observation is really useful: it implies that the function

$$\phi : \mathbb{Z} \to \pi_1(S^1, 1), \quad \phi(n) = [\omega_n]$$

is a group homomorphism. Our goal now is to show that ϕ is an isomorphism.

The intuition here is that the collection of maps $\{\omega_n\}_{n\in\mathbb{Z}}$ acts a lot like the integers, where the number $n \in \mathbb{Z}$ corresponds to going around the circle n-times counter-clockwise. The point of our calculation is that that is all that can happen.

- Showing the map ϕ is surjective says that every loop in S^1 looks like "going around the circle" some number of times.
- Showing the map φ is injective says that going around the loop n-times and going around the loop m-times is never somehow equivalent in $\pi_1(S^1, 1)$ when $n \neq m$.

The "reason" why ϕ is an isomorphism is that, locally, S^1 just looks like the real numbers $\mathbb{R}$. Since $\mathbb{R} \simeq *$, meaning $\pi_1(\mathbb{R}, 1) = 0$, there can be no weird local behavior of S^1 that obfuscates our argument.

There is a continuous map $p : \mathbb{R} \to S^1$ which will feature very heavily in these arguments. This map is defined by

$$p(t) = e^{2\pi i\cdot t},$$

which we can geometrically think of as "winding the real numbers around S^1". Consider again the map $\omega_n : I \to S^1$. We can lift this map along p to a map $\tilde{\omega}_n : I \to \mathbb{R}$ where

$$\tilde{\omega}_n(t) = n \cdot t.$$

By "lift along p", I mean that the maps ω_n and p fit into a diagram of the form:

$$\begin{array}{ccc} & & \mathbb{R} \\ & & \downarrow p \\ I & \xrightarrow{\omega_n} & S^1 \end{array}$$

and the map $\tilde{\omega}_n$ fills in the triangle and makes the diagram commute:

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\tilde{\omega}_n}{\nearrow} & \downarrow p \\ I & \xrightarrow{\omega_n} & S^1 \end{array}$$

In fact, the existence of a lift is not unique to the paths γ_n. All paths in S^1 lift to $\mathbb{R}$, and they lift *uniquely* as soon as we specify any value.

Proposition 2.10 (Path lifting)**.** *Let $f : I \to S^1$ be any path with $\gamma(0) = x \in S^1$. Let $\tilde{x} \in p^{-1}(x)$ be any point in the fiber over x. Then there is a unique lift $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}(0) = \tilde{x}$ and $f = p \circ \tilde{f}$.*

Exercise 2.11. Prove the above proposition. Notice that as there are infinitely many points in the fiber $p^{-1}(x) \subseteq \mathbb{R}$ (in fact if $t \in p^{-1}(x)$ then $t + n \in p^{-1}(x)$ for any $n \in \mathbb{Z}$), the choice of lift $\tilde{f}$ is certainly not unique unless we specify $\tilde{f}(0)$.

Path lifting is nice. Even stronger is that we can lift entire homotopies from S^1 to $\mathbb{R}$.

Proposition 2.12 (Homotopy path lifting)**.** *Let $H : I \times I \to S^1$ be a path homotopy from $H(s,0)$ and $H(s,1)$, and let $x = H(0,t)$ for any $t \in I$. Let $\tilde{x} \in p^{-1}(x)$. Then there is a unique lift $\tilde{H} : I \times I \to \mathbb{R}$ such that $H = p \circ \tilde{H}$ and $\tilde{H}(0,t) = \tilde{x}$ for all $t \in I$.*

Remark 2.13. We will not prove the above proposition. For reference, see section 1.1 of Hatcher's book.

Using these propositions, we will prove that $\pi_1(S^1,1) \cong \mathbb{Z}$.

First, we show that ϕ is surjective. Let $\gamma : (I,0) \to (S^1,1)$ be a loop. Path lifting implies that there is a unique lift $\tilde{\gamma} : (I,1) \to (\mathbb{R},0)$ such that $\tilde{\gamma}(0) = 0$.

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow p \\ I & \xrightarrow{\gamma} & S^1 \end{array}$$

By commutativity of the above diagram, we have that $p \circ \tilde{\gamma}(1) = \gamma(1) = \gamma(0) = 1$, thus $\tilde{\gamma}(1) \in p^{-1}(1)$. Since $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$, this implies that $\tilde{\gamma}(1)$ is some integer n.

There is another path in $\mathbb{R}$ from 0 to n: it is the lift $\tilde{\omega}_n : I \to \mathbb{R}$. We have a homotopy from $\tilde{\gamma}$ to $\tilde{\omega}_n$:

$$H : I \times I \to \mathbb{R}, \quad H(s,t) = (1-t)\tilde{\gamma}(s) + t \cdot \tilde{\omega}_n(s).$$

Now we can just compose with p, getting a map $p \circ H : I \times I \to \mathbb{R} \to S^1$. By construction, this map exhibits a homotopy between γ and ω_n. Hence $[\omega_n] = [\gamma]$, so ϕ is surjective.

Now, suppose that $\gamma : I \to S^1$ is some path such that $\gamma \simeq \omega_n$ and $\gamma \simeq \omega_m$. Homotopy equivalence is transitive, so there is a homotopy

$$H : I \times I \to S^1$$

from ω_n to ω_m. By Homotopy path lifting, there is a unique lift $\tilde{H} : I \times I \to \mathbb{R}$ such that $\tilde{H}(0,t) = 0$.

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\tilde{H}}{\nearrow} & \downarrow p \\ I \times I & \xrightarrow{H} & S^1 \end{array}$$

By uniqueness of path lifting, since $\tilde{H}(s,0)$ is a lift of ω_n which starts at 0 and ends at n, we must have that $\tilde{H}(s,0) = \tilde{\omega}_n(s)$. Similarly, $\tilde{H}(s,1) = \tilde{\omega}_m(s)$. Since $\tilde{H}$ is a path homotopy, we must have that $\tilde{H}(1,t)$ is a fixed point $x \in \mathbb{R}$. When $t = 0$, we see that $\tilde{H}(1,0) = \tilde{\omega}_n(1) = n$. When $t = 1$, we see that $\tilde{H}(1,1) = \tilde{\omega}_m(1) = m$. Therefore, we must have that $n = m$, hence ϕ is injective, finishing the proof.

2.2. **More properties and computations.**

Lecture 6.

Alright! We have our first nontrivial fundamental group computation. How far can we milk it? In other words, are there other spaces out there whose fundamental group we can compute with our knowledge?

To really answer this question, we should develop a few more tools for working with the fundamental group. Let (X,x_0) and (Y,y_0) be based spaces, and consider their product $(X \times Y,(x_0,y_0))$. There are continuous projection maps

$$p_X : (X \times Y,(x_0,y_0)) \to (X,x_0), \quad p_X(x,y) = x,$$
$$p_Y : (X \times Y,(x_0,y_0)) \to (Y,y_0), \quad p_Y(x,y) = y.$$

Recall from our last homework that continuous maps of based spaces induce group homomorphisms between fundamental groups. Thus, the projection maps induce group homomorphisms

$$(p_X)_* : \pi_1(X \times Y,(x_0,y_0)) \to \pi_1(X,x_0),$$
$$(p_Y)_* : \pi_1(X \times Y,(x_0,y_0)) \to \pi_1(Y,y_0).$$

Each of these induced maps just takes a loop in $X \times Y$ and postcomposes with the projection map. The point of the following proposition is that these projection maps let us reconstruct the fundamental group of the product space for the fundamental groups of its factors.

Proposition 2.14. *The group homomorphism*

$$P = ((p_X)_*, (p_Y)_*) : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

defined on loops $\gamma : I \to X \times Y$ *by*

$$P([\gamma]) = ((p_X)_*([\gamma]), (p_Y)_*([\gamma])) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is an isomorphism.

Proof. First, we show surjectivity. For any $[\varphi] \in \pi_1(X, x_0)$ and $[\psi] \in \pi_1(Y, y_0)$, define the loop $f : I \to X \times Y$ be the loop given by

$$f(t) = (\varphi(t), \psi(t)).$$

Notice that by construction, $p_X \circ f = \varphi$ and $p_Y \circ f = \psi$. Then we have

$$P([f]) = ((p_X)_*([f]), (p_Y)_*([f])) = ([p_x \circ f], [p_Y \circ f]) = ([\varphi], [\psi]).$$

This establishes surjectivity.

We next show that P is injective. Suppose that $[f] \in \pi_1(X \times Y, (x_0, y_0))$ is a loop such that $P([f]) = ([c_{x_0}], [c_{y_0}])$ (remember that the identity element of the fundamental group is the constant loop at the basepoint, so the identity element of the product is the ordered pair consisting of the constant paths at the respective basepoints). By definition, we have

$$p_X \circ f \simeq c_{x_0}, \quad p_Y \circ f \simeq c_{y_0}.$$

Let $H_1 : I \times I \to X$ be the first homotopy and $H_2 : I \times I \to Y$ the second. We can just “do these homotopies in each coordinate" to show that $[f] = [c_{(x_0, y_0)}]$. Let $H : I \times I \to X \times Y$ be defined by

$$H(s, t) = (H_1(s, t), H_2(s, t)).$$

Observe:

$$H(s, 0) = (H_1(s, 0), H_2(s, 0)) = (p_X \circ f(s), p_Y \circ f(s)) = f(s),$$

and

$$H(s, 1) = (H_1(s, 1), H_2(s, 1)) = (c_{x_0}(s), c_{y_0}(s)) = (x_0, y_0) = c_{(x_0, y_0)}(s).$$

Thus, H is a homotopy from f to $c_{(x_0, y_0)}$, so P must be injective. Therefore P is an isomorphism. □

Exercise 2.15. This generalizes to finite products of any length! Namely, the map

$$P = ((p_{X_i})_*)_{i=1}^n : \pi_1\left(\prod_{i=1}^n X_i, (x_i)_{i=1}^n\right) \to \prod_{i=1}^n \pi_1(X_i, x_i)$$

is an isomorphism of groups.

This tells us some cool stuff. For example:

- The torus $T = S^1 \times S^1$ has fundamental group

  $$\pi_1(T) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}.$$

  You can imagine the two distinct $\mathbb{Z}$'s as “going around the holes" of T in different ways. One can travel around the circumference of the torus, and one can also travel along the circle going inwards towards the center. These are the two generators for $\pi_1(T)$.
- Similarly, the n-torus $S^1 \times \cdots \times S^1$ has fundamental group $\mathbb{Z} \times \cdots \times \mathbb{Z}$.
- We have seen that the fundamental group of $\mathbb{R}$ is trivial, as is $\mathbb{R}^n$ for any n. In particular, this implies that the $\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1 \times \mathbb{R}^n) = 0$.

We showed on the homework that $M \simeq S^1$, where M denotes the Möbius strip. In fact, the fundamental group respects this homotopy equivalence!

Proposition 2.16. *Let* $f : X \to Y$ *be a homotopy equivalence. Then for any* $x_0 \in X$*, the induced map*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism.

As a result, we get two more calculations:

- If $X \simeq S^1$, then $\pi_1(S^1) \cong \mathbb{Z}$. So, spaces like the annulus or the Möbius strip have fundamental groups given by $\mathbb{Z}$.
- If $X \simeq *$, then $\pi_1(X) = 0$.

Proof. Since $f : X \to Y$ is a homotopy equivalence, there is a homotopy inverse $g : Y \to X$, i.e. we have $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. To show that f_* is an isomorphism, we would like to say that, well, g is a homotopy inverse, so g_* ought to be a group inverse as well! That is, one might guess that

$$f_* \circ g_* = \mathrm{id}_{\pi_1(Y,y_0)}, \quad g_* \circ f_* = \mathrm{id}_{\pi_1(X,x_0)}.$$

However, it is not *quite* this simple. Implicitely, we are assuming above that if $x_0 \in X$ is a chosen basepoint, that $g \circ f(x_0) = x_0$. However, this need not be the case at all. In particular, the domain and codomain of

$$g_* \circ f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \to \pi_1(X, g(f(x_0)))$$

need not be the same group. **HOWEVER...** we can surmount this obstacle.

Since $g \circ f \simeq \mathrm{id}_X$, there is a homotopy

$$H : X \times I \to X$$

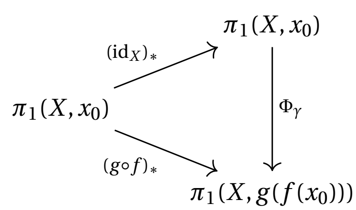
where $H(x,0) = g \circ f(x)$ and $H(x,1) = x$. In fact, (even without any assumption on path-connectedness of X!) this allows us to create a path from $g(f(x_0))$ to x_0: let $\gamma : I \to X$ be defined by

$$\gamma(t) = H(x_0, t).$$

Then $\gamma(0) = H(x_0,0) = g \circ f(x_0)$ and $\gamma(1) = H(x_0,1) = x_0$. Recall from the proof that the fundamental group of a path-connected space is independent of choice of basepoint that the map

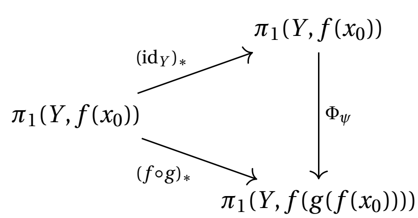
$$\Phi_\gamma : \pi_1(X, x_0) \to \pi_1(X, g(f(x_0))), \quad \Phi_\gamma([\psi]) = [\gamma] \cdot [\psi] \cdot [\gamma^{\mathrm{rev}}]$$

is an isomorphism of groups. It is an exercise in symbol pushing (a good one, if you want to try it!) to show that Φ_γ fits into a commutative diagram of the form



By the first homework assignment, we saw that $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$. In particular, it is an isomorphism. We also saw that $(g \circ f)_* = g_* \circ f_*$. Since Φ_γ is also an isomorphism, commutativity implies that $g_* \circ f_*$ is an isomorphism, thus f_* is injective (and g_* is surjective).

Alternatively, we can use the homotopy between $f \circ g \simeq \mathrm{id}_Y$ to get a path $\psi : I \to Y$ from $f(x_0)$ to $f(g(f(x_0)))$ (this is annoying and cumbersome to write but it is what it is) and form the associated map of fundamental groups Φ_ψ. Again, this fits into a commutative diagram of the form



Again, we have $(\mathrm{id}_Y)_* = \mathrm{id}_{\pi_1(Y,f(x_0))}$ and Φ_ψ are isomorphisms, so $(g \circ f)_* = g_* \circ f_*$ must also be. Therefore f_* is surjective (and g_* is injective), so f_* is an isomorphism. □

2.3. **Application: the Brouwer fixed point theorem.**

Lecture 7.

There are many more fun things to do with the fundamental group. Some of these things are on Homework 2. One of them is the next theorem, which is a real cornerstone in "using algebra to solve topological problems". Recall that $\overline{\mathbb{D}}^2 \subseteq \mathbb{R}^2$ denotes the closed unit disk.

Theorem 2.17 (**Brouwer Fixed Point Theorem**). *Let $f : \overline{\mathbb{D}}^2 \to \overline{\mathbb{D}}^2$ be any continuous map. Then there is some point $x \in \overline{\mathbb{D}}^2$ such that $f(x) = x$.*

Proof. For sake of contradiction, suppose that $f(x) \neq x$ for all points $x \in \overline{\mathbb{D}}^2$. Define a function $r : \overline{\mathbb{D}}^2 \to S^1$ by sending $x \in \overline{\mathbb{D}}^2$ to the point on S^1 at the end of the ray in $\mathbb{R}^2$ which starts at $f(x)$ and passes through x. Note that this is well-defined as two distinct points specify a line, and as $f(x) \neq x$, this ray always contains two distinct points. Notice also that if $y \in S^1 \subseteq \overline{\mathbb{D}}^2$ lies on the boundary, then this ray intersects S^1 at exactly x. In other words, $r(y) = y$. Moreover, restricting the domain of r to S^1 gives the map

$$r|_{S^1} : S^1 \to S^1,$$

and by our previous argument, we have that $r|_{S^1} = \mathrm{id}_{S^1}$.

Consider the composition of continuous functions

$$S^1 \xrightarrow{i} \overline{\mathbb{D}}^2 \xrightarrow{r} S^1.$$

This induces group homomorphisms on the corresponding fundamental groups

$$\pi_1(S^1) \xrightarrow{i_*} \pi_1(\overline{\mathbb{D}}^2) \xrightarrow{r_*} \pi_1(S^1).$$

Notice that the composition $r \circ i$ is the same as $r|_{S^1}$, which we also know is the same as id_{S^1}. Since $(\mathrm{id}_{S^1})_* \cong \mathrm{id}_{\pi_1(S^1)}$, the composition $r_* \circ i_*$ is equal to the identity map on $\pi_1(S^1) \cong \mathbb{Z}$. However, since $\overline{\mathbb{D}}^2 \simeq *$, we have that $\pi_1(\overline{\mathbb{D}}^2) = 0$. This means we can rewrite the diagram of induced maps on fundamental groups as

$$\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{r_*} \mathbb{Z}.$$

From this perspective, we must have that $r_* \circ i_*$ is the 0 map. This contradicts our previous finding, so there must be some point where $f(x) = x$. □

This ends (for now) the discussion on the fundamental group. We now move into our next topic for the course: category theory.

3. Category theory

Lecture 8.

We have seen that there are nice invariants of spaces known as homotopy groups, and have gotten our hands dirty a little bit with the first of these invariants, i.e. the fundamental group. This process takes topology and assigns to it some algebra, and it does so in a way that "preserves structure". To be precise, to any based space (X, x_0) we assigned a group $\pi_1(X, x_0)$, and to any continuous map of based spaces $f : (X, x_0) \to (Y, y_0)$ we assigned a group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Moreover, this procedure respects function composition and sends the identity map of spaces to the identity map of fundamental groups.

One of the goals of category theory is to generalize this relationship. In doing so, we create a very general, theoretical area of math that is widely applicable, serving as a language to interpret and demystify various phenomena which makes you say "huh, that is kinda similar to this other thing I saw one time..."

3.1. **Categories.** A *category* $\mathcal{C}$ consists of the following data:

- A class of *objects*, which we may denote by $\mathrm{ob}(\mathcal{C})$;
- For any two objects $X, Y \in \mathcal{C}$, a set of *morphisms* or *maps* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, i.e. $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ may be represented by some arrow $f : X \to Y$;

- A *composition* operation, meaning for any objects $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, there is a function of sets
$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
which represents composition: if $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, then $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X,Z)$ is the map you would expect: $X \xrightarrow{f} Y \xrightarrow{g} Z$. Composition satisfies two rules:
  - (*Associativity*) Whenever this expression makes sense, we have
$$(h \circ g) \circ f = h \circ (g \circ f).$$
  - (*Identity*) For all $X \in \mathrm{ob}(\mathcal{C})$, there is an identity map $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, we have
$$f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f.$$

We say that a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an *isomorphism* if there exists another morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that
$$f \circ g = \mathrm{id}_Y, \quad g \circ f = \mathrm{id}_X.$$

You should think of the objects of a category as a collection of “types of things" one does mathematics with, and the morphisms as functions which preserve the inherent structure that these objects have. Here are some familiar categories.

Example 3.1.

- The collection of all sets and all set functions defines the category $\mathcal{S}\mathrm{et}$. The isomorphisms in this category are bijections.
- The collection of all topological spaces and all continuous functions defines the category $\mathcal{T}\mathrm{op}$. The isomorphisms in this category are homeomorphisms.
- We also have a category $\mathrm{h}\mathcal{T}\mathrm{op}$ with the same objects, but with morphisms given by homotopy classes of continuous functions. The isomorphisms in this category are homotopy equivalences.
- The collection of all based topological spaces and all basepoint preserving continuous functions defines the category $\mathcal{T}\mathrm{op}_*$. The isomorphisms are based homeomorphisms. Similarly to above, we also have a category $\mathrm{h}\mathcal{T}\mathrm{op}_*$. The isomorphisms are based path-homotopy equivalences.
- The collection of all vector spaces over a field k and all linear transformations defines the category $\mathcal{V}\mathrm{ect}_k$. The isomorphisms are vector space isomorphisms.
- The collection of all R-modules over a commutative ring R with R-linear maps defines the category $\mathrm{Mod}(R)$. The isomorphisms are module isomorphisms.
- The collection of all groups and group homomorphisms defines the category $\mathcal{G}\mathrm{rp}$. The isomorphisms are group isomorphisms.
- The collection of all abelian groups and group homomoprhism defines the category $\mathcal{A}\mathrm{b}$. The isomorphisms are group isomorphisms.

However, the definition of a category is exceptionally general. While our first intuition for a category produces the ones listed above, there are many, many more examples. Here are some more exotic categories.

Example 3.2.

- Let $(P, \leq)$ be a poset (so that $\leq$ is a reflexive, antisymmetric, transitive binary relation). We may regard P as a category, which we will denote $P_{\leq}$, by letting every point $x \in P$ be an object $x \in \mathrm{ob}(P_{\leq})$, and for every relation $x \leq y$ in P, assigning a unique morphism $x \to y$ in $P_{\leq}$. Since $x \leq x$, we have a unique morphism $x \to x$ (the identity!). Can you see how to get the other requirements to be a category from the poset relations?
- Let k be a field. There is a category $\mathcal{M}\mathrm{at}_k$ whose objects are nonnegative integers $bnin\mathbb{N}$ and where a morphism $n \to m$ is represented by an $m \times n$-matrix with entries in k. Composition is given by matrix multiplication.
- If $\mathcal{C}$ is any category, then its *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, but for every morphism $f : x \to y$ in $\mathcal{C}$, we have a morphism $f^{\mathrm{op}} : y \to x$ in $\mathcal{C}^{\mathrm{op}}$.

- Let G be a group. Then we can construct a category, sometimes denoted BG or even just G, with only one object $*$ and where $\mathrm{Hom}_{BG}(*,*) = G$. How does composition work? Well, we know how to multiply elements in a group, right?
- There is a category with two non-isomorphic objects between them and no morphisms between these objects. Here is a way to represent this category, where we suppress the isomorphisms:

  $$\bullet \quad \bullet$$

  There is a variant of this category where there is a single morphisms between our objects, which we can represent as

  $$\bullet \to \bullet$$

  Or, say we had two distinct morphisms between our objects, which can be represented as

  $$\bullet \rightrightarrows \bullet$$

  Or how about a category with three objects, and two morphisms that fit into the following diagram:

  $$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

  Or how about a category with three objects, and two morphisms going the other way?

  $$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

Exercise 3.3. What are the isomorphisms in these funky categories?

The 4^{th} example fits into a class of categories called *groupoids*. A category is a groupoid if every morphism is an isomorphism. Another example of a groupoid which we will see in Homework 2 is called the *fundamental groupoid*. If X is a topological space, then the fundamental groupoid of X, denoted $\Pi_1(X)$, is the category where:

- The objects of $\Pi_1(X)$ are the points of X, and
- For any two points $x, y \in X$, regarded as objects in $\Pi_1(X)$, a morphism $f \in \mathrm{Hom}_{\Pi_1(X)}(x,y)$ is given by a homotopy class of paths.

We will soon see how this category captures the data of all of the different fundamental groups of X for different choices of basepoints.

Lecture 9.

3.2. **Functors.** We can also transport information between categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories consists of the following data:

- for every object $c \in \mathrm{ob}(\mathcal{C})$, an object $F(c) \in \mathrm{ob}(\mathcal{D})$;
- for every morphism $f : x \to y$ in $\mathcal{C}$, a morphism $F(f) : F(x) \to F(y)$ in $\mathcal{D}$.

A functor is required to satisfy the following:

- if $\mathrm{id}_c : c \to c$ is the identity function for an object $c \in \mathcal{C}$, then $F(\mathrm{id}_c) = \mathrm{id}_{F(c)} : F(c) \to F(c)$ is the identity function for $F(c)$;
- if $g \circ f : x \to y \to z$ is defined, then $F(g \circ f) = F(g) \circ F(f) : F(x) \to F(y) \to F(z)$.

Example 3.4.

- If $\mathcal{C}$ is a category which has objects “sets with some structure" (such as topological space, groups, vector spaces, ...) and morphisms “set functions which preserve this structure" (such as continuous functions, group homomorphisms, linear transformations, ...), then there is a *forgetful functor* $U : \mathcal{C} \to \mathcal{S}\mathrm{et}$, where $U(c) \in \mathrm{ob}(\mathcal{C})$ is the object c just viewed as a set, and where any morphism $f : x \to y$ in $\mathcal{C}$ is sent to its underlying morphism of sets.

- By homework 1, we have seen that the fundamental group defines a functor
$$\pi_1 : \mathcal{T}\mathrm{op}_* \to \mathcal{G}\mathrm{rp}$$
from based topological spaces to groups. Moreover, since π_1 respects homotopy equivalence, this actually gives a functor
$$\pi_1 : \mathrm{h}\mathcal{T}\mathrm{op} \to \mathcal{G}\mathrm{rp}.$$
- There is a functor
$$(-)^* : \mathcal{V}\mathrm{ect}_k \to \mathcal{V}\mathrm{ect}_k$$
which takes a k-vector space V to its linear dual $V^* = \mathrm{Hom}_k(V, k)$. This is an example of an *endofunctor*, a functor from a category to itself.
- For any category $\mathcal{C}$ and any object $X \in \mathrm{ob}(\mathcal{C})$, there are two functors from $\mathcal{C}$ which land in set which X determines. These are given by considering maps in $\mathcal{C}$ either into X or out of X. The first is
$$\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \to \mathcal{S}\mathrm{et}, \quad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X).$$
Notice that if $f : Y \to Z$ is a morphism in $\mathcal{C}$, then
$$\mathrm{Hom}(f, X) : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$$
is given by precomposition with f and flips the order of Y and Z! A functor F with this property, one which sends $f : X \to Y$ to $F(f) : F(Y) \to F(X)$ is called *contravariant*. Notice as well that a contravariant functor out of $\mathcal{C}$ is the same as a covariant functor out of $\mathcal{C}^{\mathrm{op}}$. Often, when people want to specify that a functor is contravariant, they will simply denote their source category by $\mathcal{C}^{\mathrm{op}}$.

  The second functor is
$$\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}\mathrm{et}, \quad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y).$$
Notice that if $f : X \to Y$ is a morphism in $\mathcal{C}$, then
$$\mathrm{Hom}(X, f) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
is given by postcomposition with f, and so this Hom functor is covariant (read: not contravariant).
- The above two functors are examples of *representable functors*. A functor $F : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is representable if there is some object $X \in \mathrm{ob}(\mathcal{C})$ such that
$$F(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X).$$
In this case, we say that X is the *representing object*. In other words, applying F is the same as mapping *into* X.

  Dually, we say that $G : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is corepresentable if there is some object $Z \in \mathrm{ob}(\mathcal{C})$ suh that
$$G(Y) = \mathrm{Hom}_{\mathcal{C}}(Z, Y).$$
In this case, we say that Y is the corepresenting object. In other words, applying G is the same as mapping *out of* Y.

  We have seen another example of a corepresentable functor: the fundamental group! This is a functor
$$\pi_1 : \mathrm{h}\mathcal{T}\mathrm{op}_* \to \mathcal{G}\mathrm{rp}$$
given by $\pi_1(X, x_0) = \mathrm{Hom}_{\mathrm{h}\mathcal{T}\mathrm{op}_*}((S^1, 1), (X, x_0))$. This is certainly corepresentable as a functor to $\mathcal{S}\mathrm{et}$, and the whole point is that not only is the set of homotopy classes of maps from the circle a set, it is also a group. Thus, we have the functor in question.

  Another way to phrase this last bit: we know there is a forgetful functor $U : \mathcal{G}\mathrm{rp} \to \mathcal{S}\mathrm{et}$ which takes a group to its underlying set. What we have said now is that the fundamental group functor *lifts* along U to $\mathcal{G}\mathrm{rp}$:
$$\begin{array}{ccc} & & \mathcal{G}\mathrm{rp} \\ & \overset{\pi_1}{\nearrow} & \downarrow U \\ \mathrm{h}\mathcal{T}\mathrm{op}_* & \xrightarrow{\pi_1} & \mathcal{S}\mathrm{et} \end{array}$$

Lecture 11.

3.3. **Natural transformations.** We can go further and define morphisms between morphisms! Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\alpha : F \implies G$ is the data of:

- a morphism $\alpha_c : F(c) \to G(c)$ in $\mathcal{D}$ for every object $c \in \text{ob}(\mathcal{C})$
- if $f : c \to c'$ is any morphism in $\mathcal{C}$, then the following diagram commutes:

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(c') \\ \alpha_c \downarrow & & \alpha_{c'} \downarrow \\ G(c) & \xrightarrow{G(f)} & G(c') \end{array}$$

Example 3.5.

- There is a functor $(-)^{**} : \text{Vect}_k \to \text{Vect}_k$ which sends a vector space to its double dual and a linear map to what you'd expect. There is a natural transformation $\text{ev} : \text{Id}_{\text{Vect}_k} \implies (-)^{**}$ whose component morphisms are evaluation! Meaning, remember that
$$V^{**} = \text{Hom}_k(\text{Hom}_k(V, k), k),$$
so that any element $f \in V^{**}$ represents a linear function
$$f : \text{Hom}_k(V, k) \to k.$$
For any vector space $V \in \text{ob}(\mathcal{V}\text{ect}_k)$, the component morphism
$$\text{ev}_V : V \to V^{**}$$
is defined by sending any vector $v \in V$ to the linear map of the same name $\text{ev}_V : \text{Hom}_k(V, k) \to k$, defined by evaluating at v. In other words, if $T \in \text{Hom}_k(V, k)$, then
$$\text{ev}_V(T) = T(v) \in k.$$
This is a natural transformation. Check the morphisms condition if you aren't convinced! This is an example of a natural isomorphism, where the component morphisms are isomorphisms.
- Let X and Y be topological spaces, and consider their fundamental groupoids $\Pi_1(X)$ and $\Pi_1(Y)$. If $f : X \to Y$ is any continuous function, then there is a functor
$$\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y), \quad x \mapsto f(x), ([\gamma] : x_1 \to x_2) \mapsto ([f \circ \gamma] : f(x_1) \mapsto f(x_2)).$$
Notice that a natural transformation behaves like a "function between functions". In the world of topology, we have another type of "function between functions": this is *exactly* what a homotopy is! And, at the level of fundamental groupoids, if $H : X \times I \to Y$ is a homotopy between maps $f, g : X \to Y$, then indeed there is a natural transformation
$$H : \Pi_1(f) \implies \Pi_1(g).$$

Remark 3.6. Let $\mathcal{C}, \mathcal{D}$ be any two categories. Then there is a category $\text{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations.

3.4. **Yoneda Lemma.** The Yoneda Lemma is a powerful tool in category theory that relates functors to each other. It is kind of a generalization of Cayley's theorem for groups, which states that every group is a subgroup of some symmetric group. Here is a general way to state it.

Lemma 3.7 (Yoneda)**.** *Let $F : \mathcal{C} \to \mathcal{S}\text{et}$ be a functor. For any object $c \in \mathcal{C}$, there is a bijection of sets*
$$\text{Nat}(\text{Hom}_{\mathcal{C}}(c, -), F) \cong F(c)$$
between the set of natural transformations from the functor $\text{Hom}_{\mathcal{C}}(c, -)$ and F and the elements of the set $F(c)$. This bijection associates a natural transformation $\alpha : \text{Hom}(c, -) \implies F$ to the element $\alpha(\text{id} : c \to c) \in F(c)$.

Think about this in terms of linear algebra. Let V be a 1-dimensional vector space and W be any vector space. A linear transformation $T : V \to W$ is uniquely determined by its value on a basis of V. Any such basis consists of exactly one nonzero vector $v \in V$, meaning that the linear transformation is uniquely determined by some value $T(v) \in W$. In other words, there is a bijection
$$\text{Hom}_{\mathcal{V}\text{ect}}(V, W) = U(W),$$

where $U : \mathcal{V}\mathrm{ect} \to \mathcal{S}\mathrm{et}$ is the forgetful functor, meaning $U(W)$ is just the vector space W considered as a set.

Here are some consequences of the Yoneda Lemma that are particularly useful.

- Let $F, G : \mathcal{C} \to \mathcal{S}\mathrm{et}$ be representable functors. So, $F = \mathrm{Hom}_{\mathcal{C}}(-, X)$ and $G = \mathrm{Hom}_{\mathcal{C}}(-, Y)$ for some objects $X, Y \in \mathrm{ob}(\mathcal{C})$. Then there is a bijection of sets:

$$\mathrm{Nat}(F, G) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

  In other words, natural transformations between representable functors are the same as maps between their representing objects.
- For any category $\mathcal{C}$, there is a functor

$$y : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$$

  which takes every object $X \in \mathrm{ob}(\mathcal{C})$ to the representable functor $\mathrm{Hom}_{\mathcal{C}}(, X)$. This functor is often called the *Yoneda embedding*. It is full and faithful by the Yoneda Lemma. In this way, the *presheaf category* $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ is some large, universal place where all categories embed, just as all groups embed into a symmetric group. In the language of the next section, the Yoneda embedding can be seen as the universal category into which $\mathcal{C}$ embeds such that all diagrams admit colimits.

Try messing around with the Yoneda lemma and the forgetful functor from a category like $\mathcal{V}\mathrm{ect}_k$, $\mathcal{G}\mathrm{rp}$, $\mathcal{T}\mathrm{op}$ to $\mathcal{S}\mathrm{et}$ and see what you can derive!

Lecture 12.

3.5. **Limits and Colimits.** Another useful way to use category theory is to make certain constructions universal or canonical. Almost all of these constructions (products, coproducts, quotients, mapping spaces, tensor products, etc) are examples of limits or colimits of particular functors.

Let $F : I \to \mathcal{C}$ be a functor (I'm writing I for a category here because I want to think of it as some type of "indexing category", like when you take an infinite sum or product over an indexing set I). We will call this a *diagram of shape I in $\mathcal{C}$*. For any morphism $f \in \mathrm{Hom}_I(X, Y)$, there is a corresponding morphism

$$F(f) : F(X) \to F(Y).$$

There are two natural things one might want to do: look at maps of objects of $\mathcal{C}$ *into* $F(f)$, and look at maps of objects *out of* $F(f)$. So, we may want to look at objects $Z \in \mathcal{C}$ "living over $F(f)$"

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ F(X) & \xrightarrow{F(f)} & F(Y) \end{array}$$

or we may want to look at objects "living under $F(f)$"

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \searrow & & \swarrow \\ & Z & \end{array}$$

To be "universal", we really want to consider the objects we which live over or under **ALL** morphisms in the diagram in $\mathcal{C}$ induced by F.

A *cone* over the functor F is the data of:

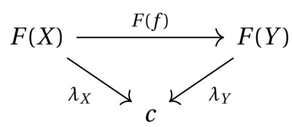
- an object $c \in \mathcal{C}$,
- a collection of morphisms $\{\lambda_X : c \to F(X)\}$ out of c for all $X \in \mathrm{ob}(I)$, such that
- for every morphism $f \in \mathrm{Hom}_I(X, Y)$ between any two objects in I, the diagram

$$\begin{array}{ccc} & c & \\ {}^{\lambda_X}\swarrow & & \searrow^{\lambda_Y} \\ F(X) & \xrightarrow{F(f)} & F(Y) \end{array}$$

  commutes.

Dually, a *cocone* over the functor F is the data of:

- an object $d \in \mathcal{C}$,
- a collection of morphisms $\{\lambda_X : F(X) \to c\}$ into c for all $X \in \mathrm{ob}(I)$, such that
- for every morphism $f \in \mathrm{Hom}_I(X, Y)$ between any two objects in I, the diagram

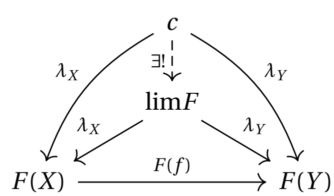


commutes.

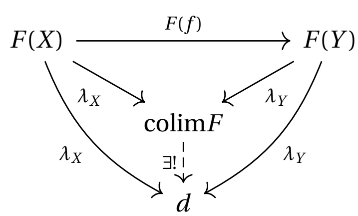
I will often refer to the objects in the "top" and "bottom" of a cone or cocone as *defining objects*.

Cones have maps into diagrams, and cocones have maps our of diagrams. Limits and colimits are universal cones and cocones, i.e. limits are the "last thing" which maps into a diagram, and colimits are the "first thing" which a diagram maps to. By first and last, I just means they are literally closest to the diagram.

To be more precise, the *limit* of a functor $F : I \to \mathcal{C}$ is a cone over F which every other cone uniquely maps to. We often denote the object in the limit by $\lim F \in \mathrm{ob}(\mathcal{C})$. Diagrammatically, the universal property of the limit is that for any other cone over F with defining object c, there is a unique morphism $c \to \lim F$ making the following diagram commute for every $f \in \mathrm{Hom}_I(X, Y)$:



Dually, the *colimit* of a functor $F : I \to \mathcal{C}$ is a cocone under F which maps to every other cocone uniquely. We often denote the object in the colimit by $\mathrm{colim} F \in \mathrm{ob}(\mathcal{C})$. Diagrammatically, the universal property of the colimit is that for any other cocone under F with defining object d, there is a unique morphism $\mathrm{colim} F \to d$ making the following diagram commute for every $f \in \mathrm{Hom}_I(X, Y)$:



If we are clever with our domain category I (this is one of the whole reasons we deal with those funny categories with finitely many objects!), one can recast many familiar constructions as either a limit or colimit. Better yet, this gives us a universal property for free!

⚠**Warning** ⚠ **3.8.** Limits and colimits need not exist! It is certainly possible that there may be a diagram $F : I \to \mathcal{C}$ with no limit or colimit. However, the categories $\mathcal{T}\mathrm{op}$, $\mathcal{T}\mathrm{op}_*$, $\mathcal{S}\mathrm{et}$, $\mathcal{A}\mathrm{b}$ all have limits and colimits, so we don't need to be too worried about this.

Lecture 13.

Example 3.9.

- Consider the category I with two distinct objects $\bullet_1$ and $\bullet_2$ such that $\mathrm{Hom}_I(\bullet_1, \bullet_2) =$, i.e. there are no morphisms between the two distinct objects in I. A diagram of shape I in $\mathcal{C}$, i.e. a functor $F : I \to \mathcal{C}$, looks like the following:

$$F(\bullet_1) \quad F(\bullet_2).$$

In other words, we are just picking out two objects in $\mathcal{C}$. The limit of this diagram is more commonly refered to as a *product*. It is some object in $\mathcal{C}$ equipped with "projection maps"

$$\begin{array}{ccc} & \lim F & \\ \swarrow & & \searrow \\ F(\bullet_1) & & F(\bullet_2) \end{array}$$

Moreover, the universal property of the limit says that if c is any any other object of $\mathcal{C}$ which also maps to $F(\bullet_1)$ and $F(\bullet_2)$, then it must factor through $\lim F$.

- If $\mathcal{C} = \mathcal{S}\text{et}$, then $\lim F$ is the cartesian product $F(\bullet_1) \times F(\bullet_2)$.
- If $\mathcal{C} = \mathcal{T}\text{op}$, then $\lim F$ is the set $F(\bullet_1) \times F(\bullet_2)$ with the product topology! Another way to think about the limit property is that $\lim_F$ must have the *coarsest* topology amongst those spaces which are cones over F. Note that if we instead worked with $\mathcal{T}\text{op}_*$, then the same space serves as the product.

On the other hand, the colimit of this diagram is more commonly referred to as a *coproduct*.

- If $\mathcal{C} = \mathcal{T}\text{op}$, then the coproduct is just the disjoint union.
- If $\mathcal{C} = \mathcal{T}\text{op}_*$, then the coproduct is *not* the disjoint union! Rather, it is what is known as the wedge sum. This is the quotient space

$$X \vee Y := X \amalg Y / (x_0 \sim y_0).$$

- The colimit of a diagram of shape

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

is called a *pushout*. In $\mathcal{T}\text{op}$, if the diagram takes the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \\ Y & & \end{array}$$

then the pushout is often denoted $X \cup_A Y$. You can think of the pushout as a recipe for how to glue X onto Y along A, with the recipe given by the image of $A \to X$. If a commutative square is a pushout square, we often denote it by

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & X \cup_A Y \end{array}$$

If A and B are sets and $f : X \to A$ and $g : X \to B$ are functions of sets, then their pushout can be identified as

$$A \cup_X B := A \amalg B / (f(x) \sim g(x)).$$

In other words, we glue A and B together at precisely the identifications made by f and g!

- We can construct the wedge sum $X \vee Y$ just as topological spaces (forgetting the basepoint) via the pushout

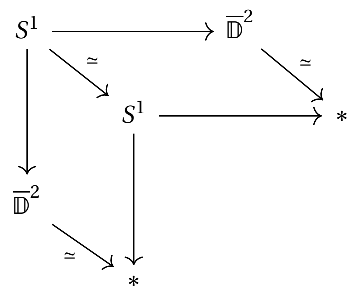
$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \\ Y & & \end{array}$$

- The pushout of the diagram of spaces

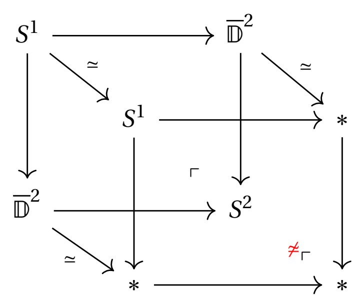
$$\begin{array}{ccc} S^1 & \longrightarrow & \overline{\mathbb{D}}^2 \\ \downarrow & & \\ \overline{\mathbb{D}}^2 & & \end{array}$$

where the component maps include along the boundary is homeomorphic to S^2! This generalizes to give a colimit construction of all spheres.

⚠**Warning** ⚠ **3.10.** One needs to be very clear about what category the colimit is taken in. Take the above example, where we formed S^2 as the colimit of a diagram in $\mathcal{T}\text{op}$. We know that $\overline{\mathbb{D}}^2$ is contractible, meaning there is a homotopy equivalence of diagrams



We computed the pushout of the outermost square to be S^2. However, the pushout of the innermost square just attaches the space $*$ (at the bottom of the diagram) to the space $*$ (at the top of the diagram) via the identifications of the map $* \to S^1$. But this map is necessarily constant: there's just not room for anything to happen! So, the pushout of the innermost diagram is again $*$. The punchline is that $S^2 \not\simeq *$.



This is illustrating that even though spaces in a diagram may be homotopy equivalent, the pushouts of these diagrams *taken in* $\mathcal{T}\text{op}$ need not be homotopy equivalent. If we wanted such a quality, we would need to compute these colimits in the homotopy category $\text{h}\mathcal{T}\text{op}$.

– The pushout of the diagram of spaces

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \\ * & & \end{array}$$

is known as the *cofiber* (or, also as the *quotient*) and usually denoted X/A.

Lecture 14.

Let's look at some diagrams and try to guess their limits/colimits!

Consider the diagram of abelian groups

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \cdots.$$

What is the colimit? Well, if d is any cocone under our diagram, then the following diagram commutes.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "1"] \arrow[d, "\lambda_0"'] & \mathbb{Z} \arrow[r, "2"] \arrow[dl, "\lambda_1"] & \mathbb{Z} \arrow[r, "3"] \arrow[dll, "\lambda_2"] & \mathbb{Z} \arrow[r, "4"] \arrow[dlll, "\lambda_3"] & \cdots \\ d \end{tikzcd}$$

We can use the structure morphisms to deduce some properties of d, and we are helped out a lot that these morphisms are all group homomorphisms: they are all determined by their value on 1! Since $\lambda_0(1) = \lambda_1(1 \cdot 1)$, and since the diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "1"] \arrow[d, "\mathrm{id}"'] & \mathbb{Z} \arrow[dl, "\mathrm{id}"] \\ \mathbb{Z} \end{tikzcd}$$

commutes, this tells us that d has to contain a copy of $\mathbb{Z}$ generated by $\lambda_0(1)$. Moving along the diagram, we have that $\lambda_0(1) = \lambda_2(2 \cdot 1 \cdot 1) = \lambda_2(2)$. Since these are group homomorphisms, we have that

$$\lambda_0(1) = \lambda_2(2) = \lambda_2(1+1) = \lambda_2(1) + \lambda_2(1) = 2 \cdot \lambda_2(1).$$

This implies that there is some element in d which is “one half" of $\lambda_0(1)$. Moving even further, we have that

$$\lambda_0(1) = \lambda_3(3 \cdot 2 \cdot 1 \cdot 1) = \lambda_3(6) = 6 \cdot \lambda_3(1).$$

In other words, we have an element $2 \cdot \lambda_3(1)$ in d which acts as a third of $\lambda_0(1)$!

Iterating this process along the diagram, we see that we can actually obtain the denominators $1/n$ for any $n \in \mathbb{Z} \setminus \{0\}$. The universal abelian group with this process, i.e. the place where we have fractions, is $\mathbb{Q}$! This is the colimit.

For our next example, let $X \in \mathrm{ob}(\mathcal{T}\mathrm{op})$ be any space. The *unbased cone* on X, denoted $C^{(u)}(X)$, is defined as the space

$$C^{(u)}(X) = X \times I / (X \times \{0\}).$$

In other words, we take the cylinder on X and crush the bottom end to a point (this is a pushout!!). The question, then, is to determine the colimit of the diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & C^{(u)}(S^1) \\ C^{(u)}(S^1) \end{tikzcd}$$

where the maps $S^1 \to C^{(u)}(S^1)$ include at $S^1 \times \{1\} \subset C^{(u)}(S^1)$ at the base of the cone.

This colimit is more or less straightforwards. As a set, it is the disjoint union of two cones which have been glued together at the images of the inclusions. In other words, it is

$$C^{(u)}(X^1) \amalg C^{(u)}(S^1) / (X \times \{1\} \amalg X \times \{1\}).$$

This is homeomorphic to S^2! One way to see this is to draw a picture: another is to notice that $C^{(u)}(S^1) \cong \overline{\mathbb{D}}^2$, and the inclusion map in the given pushout is the same as including into the boundary of $\overline{\mathbb{D}}^2$. We computed this pushout last class!

Finally, in abelian groups, what is the limit of the following?

$$\begin{tikzcd} & \mathbb{Z} \arrow[d, "n"] \\ \mathbb{Z} \arrow[r, "m"'] & \mathbb{Z} \end{tikzcd}$$

As an object, certainly the objects in the limit are pairs of integers $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ which make the diagram commute, i.e. $mx = ny$. Notice that for any choice of x, so long as one of m and n are nonzero, we have that

$y = mx/n$. In other words, a choice of one integer in (x, y) determines the other. This implies that the object of the limit is $\mathbb{Z}$. As for the morphisms, I claim that the completed limit diagram takes the form

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{b} & \mathbb{Z} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle n} \\ \mathbb{Z} & \xrightarrow[m]{} & \mathbb{Z} \end{array}$$

where $am = bn = \mathrm{lcm}(x, y)$. Think about how these particular morphisms guarantee that not only have we defined a cone over the diagram, but that it necessarily must be the limit!

Lecture 15.

Let's rebase one more time and ground ourselves by giving some formulas for computing limits and colimits in $\mathcal{S}$et. Something we will take for granted, and something that is particularly useful and makes everything work, is the fact that $\mathcal{S}$et is complete and cocomplete, meaning it has all limits and colimits.

Here are some formulas.

- Let I be the category with n distinct objects named $\{1, 2, ..., n\}$ and morphisms

$$\mathrm{Hom}_I(m, n) = \begin{cases} \varnothing & m \neq n \\ \{\mathrm{id}\} & m = n. \end{cases}$$

  Thus, there are identity morphisms for each object and that's it. A functor $F : I \to \mathcal{S}\mathrm{et}$ is just a collection of objects $\{X_j = F(j) : 1 \leq j \leq n\}$, which we might organize as

$$X_1 \quad X_2 \quad \cdots \quad X_{n-1} \quad X_n.$$

  If c is a cone over this diagram, then there are morphisms $c \to X_j$ for each j. Thus, we can think about c as a tuple $(x_1, x_2, \ldots, x_n)$ of elements $x_j \in X_j$. But there is a universal such object! Notice that there is a map of sets

$$\{(x_1, x_2, \ldots, x_n)\} \to \{(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) : x_{i_j} \in X_j\}.$$

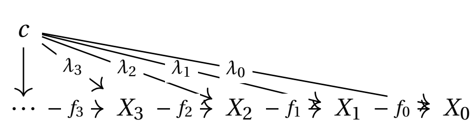
  This second set is the collection of all tuples one can possible make out of $X_1, X_2, \ldots, X_n$. Another word for this object is the product $\prod_{i=1}^n X_i$.

  Dually, if d is a cocone under this diagram, then d must consists of *disjoint* points $x_1 \amalg x_2 \amalg \cdots \amalg x_n$. The converse of the limit diagram shows that the the colimit must be the disjoint union, or coproduct, $\coprod_{i=1}^n X_i$.

  This is something that I think is very useful: if I have any indexing category I and any diagram of shape I in $\mathcal{C}$, then something we can do is just forget that I had any morphisms (other than identity ones), and this example identifies the limit or colimit as the product or coproduct. When we add back the morphisms in our diagram, we must alter the product or coproduct in *exactly* the way dictated by the commutativity of the ensuing diagrams. In other words, one can think of limits and colimits as alterations of products and coproducts, whose alterations are dictated by the morphisms in our diagram.
- Consider a diagram

$$\cdots \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0.$$

  This is a variant of the previous diagram, and we can think of the limit as a variant of the limit. Certainly without the morphisms $X_{n+1} \to X_n$, the product $\prod X_i$ maps into this diagram. However, any cone must make all of the triangles in the following diagram commute



  This commutativity condition *defines* the limit! I claim that

$$\lim F = \{(x_i) \in \prod X_i \,|\, f_n(x_{n+1}) = x_n\}.$$

In other words, the limit consists of those elements of the product, also known as sequences, are connected by the morphisms f_i. Notice: this is a *subset* of the product.

- Dually, I claim that the colimit of

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

is a variant of the coproduct. It is given by

$$\mathrm{colim} F = \coprod X_i / \sim$$

where if $x_i \in X_i$ and $x_j \in X_j$, then we have the relation $x_i \sim x_j$ if they are "eventually identified", meaning that for some n and m, we have

$$f_{j+n} \circ \cdots \circ f_j(x_j) = f_{i+m} \circ \cdots \circ f_i(x_i).$$

Notice: this is a *quotient* of the coproduct.

- I'll run through the next two. We have already talked about pushouts, which are colimits of a diagram taking the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array}$$

In terms of sets, the pushout is defined as a quotient of the coproduct $Y \amalg Z$ at exactly the identifications determined by f and g! Namely, if $x \in X$, then we we identify the points $f(x) \sim g(x)$ in the coproduct:

$$\mathrm{colim} F = Y \cup_X Z = Y \amalg Z / (f(x) \sim g(x)\ \forall x \in X).$$

Dual to this, and our last example of a particular diagram we are interested in, is the limit of this type of diagram:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & B \end{array}$$

The limit of this type of diagram is called a *pullback* or *fiber product*. It is a subset of the product $X \times Y$ consisting of those tuples (x, y) which get identified in B, meaning $f(x) = g(y)$:

$$\lim F = X \times_B Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

If $U, V \subseteq X$ are subspaces of some topological space X, then the pullback

$$\begin{array}{ccc} & & U \\ & & \downarrow{\scriptstyle \iota_U} \\ V & \xrightarrow{\iota_V} & X \end{array}$$

where the maps are the obvious inclusions, is defined as

$$U \times_X V = \{(x, y) \in U \times V : \iota_U(x) = \iota_V(y) \in X\}$$

I claim now that the map

$$U \times_X V \to U \cap V, \quad (x, y) \mapsto \iota_U(x)$$

is a homeomorphism (notice that the maps ι_U and ι_V aren't really doing anything!). I won't prove this; you should!

The pullback of a diagram of spaces

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ * & \longrightarrow & B \end{array}$$

is known as the *fiber* and usually denoted X_b. It is not hard to see why this notation is used: the map $* \to B$ picks out some point $b \in B$, and if we label the map $f : X \to B$, then $X_b = f^{-1}(b)$.

Fibers and cofibers turn out to be very very important in homotopy theory. Before turning to them, we discuss a special class of topological spaces.

Lecture 16.

3.6. **CW complexes.** As some motivation, let's remember our nice way to build S^2 via pushout squares:

$$\begin{array}{ccc} S^1 & \longrightarrow & \overline{\mathbb{D}}^2 \\ \downarrow & \ulcorner & \downarrow \\ \overline{\mathbb{D}}^2 & \longrightarrow & S^2 \end{array}$$

Here, the top two maps are inclusion into the boundary. This pushout glues two discs $\overline{\mathbb{D}}^2$ two each other along their boundary circles. We can imitate this process to now construct S^3. Notice that the boundary of $\overline{\mathbb{D}}^3$ is homeomorphic to S^2, hence there is an inclusion $S^2 \to \overline{\mathbb{D}}^3$. By the same logic as before, we can construct S^3 as a pushout:

$$\begin{array}{ccc} S^2 & \longrightarrow & \overline{\mathbb{D}}^3 \\ \downarrow & \ulcorner & \downarrow \\ \overline{\mathbb{D}}^3 & \longrightarrow & S^3 \end{array}$$

Going down in dimension, we also could have constructed S^1 as a pushout:

$$\begin{array}{ccc} S^0 & \longrightarrow & \overline{\mathbb{D}}^1 \\ \downarrow & \ulcorner & \downarrow \\ \overline{\mathbb{D}}^1 & \longrightarrow & S^1 \end{array}$$

This gives us an inductive process to construct any sphere S^n by "attaching spheres" to S^{n-1}. Moreover, we have done this in such a way that we can view any smaller dimensional sphere S^{n-k} as a particularly nice subspace: we just forget about the spheres in dimensions greater than $n-k$.

This is the motivation for our next definition. A *CW-complex* is a topological space X which is the colimit of a diagram $F : \mathbb{Z}_{\geq -1} \to \mathcal{T}\mathrm{op}$, i.e., letting $X_i = F(i)$,

$$X = \operatorname{colim}(\varnothing = X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots),$$

where this diagram has the property that X_n is the result of "attaching n-cells" to X_{n-1}. In other words, for each n, there is a pushout diagram

$$\begin{array}{ccc} \coprod S^{n-1} & \longrightarrow & \coprod \overline{\mathbb{D}}^n \\ \downarrow & \ulcorner & \downarrow \\ X_{n-1} & \longrightarrow & X_n \end{array}$$

This definition tells us that there is kind of an algorithm for constructing a CW-complex.

- Start with the 0-cells $X_0 = \coprod \overline{\mathbb{D}^0} = \coprod *$. These are just points!
- There is a pushout diagram

$$\begin{array}{ccc} \coprod S^0 & \longrightarrow & \coprod \overline{\mathbb{D}}^1 \\ \downarrow & \ulcorner & \downarrow \\ X_0 & \longrightarrow & X_1 \end{array}$$

  which creates paths between specified points in X_0. Think: a map $S^0 \to X_0$ just chooses two points in X_0, and this pushout just glues the ends of the interval $\overline{\mathbb{D}}^1 = I$ to those points!

- There is a pushout diagram

$$\begin{array}{ccc} \coprod S^1 & \longrightarrow & \coprod \overline{\mathbb{D}}^2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

which creates “surfaces" in X_1. Think: a map $S^1 \to X_1$ is a map into a space which is currently just a collection of points and some line segments between them. The types of maps we can have are
  - constant at some 0-cell $S^1 \to x_0 \in X_1$, creating an S^2;
  - if we have 0-cells $x_1, \ldots, x_n$ and have 1-cells $x_0 \to x_1, x_1 \mapsto x_2, \ldots x_{n-1} \mapsto x_n$, and $x_n \mapsto x_1$, then we can have a map $S^1 \to X$ which “fills in this loop"!
  - etc etc

I will call X_k, the union of all cells of dimensions $\leq k$, the *k-skeleton* of X.

Let’s look at some examples of spaces which can be realized as CW-complexes.

Example 3.11.

- Consider the unit interval $I = [0,1]$. We can build this in an obvious way as a CW complex. Take $X_0 = \{*, *\}$ to be two disjoint points. Attach a 1-cell to X_0 by mapping the two points of S^0 to the two points of X_0:

$$\begin{array}{ccc} S^0 & \longrightarrow & \overline{\mathbb{D}}^1 = I \\ \downarrow & & \downarrow \\ X_0 = \{*, *\} & \longrightarrow & X_1 \end{array}$$

We see already that $X_1 \cong I$. So, just don’t attach anymore cells! this forces $X_k = X_1$ for all $k \geq 1$, and so the colimit of

$$X_0 = \{*, *\} \to X_1 = I \to X_2 = I \to X_3 = I \to \cdots$$

is just I.

- Consider the real line $\mathbb{R}$. Take $X_0 = \mathbb{Z}$ to be the discrete set of integers in $\mathbb{R}$. We can attach 1-cells via maps $f_i : S^0 \to \{i, i+1\}$ for $i \in \mathbb{Z}$, so that our pushout

$$\begin{array}{ccc} \coprod_{i\in\mathbb{Z}} S^0 & \longrightarrow & \coprod_{i\in\mathbb{Z}} \overline{\mathbb{D}}^1 \\ {\scriptstyle \amalg f_i}\downarrow & & \downarrow \\ X_0 & \longrightarrow & X_1 \end{array}$$

glues in 1-cells between the integers, filling out the space in $\mathbb{R}$. This is all we need to do!

- Real projective space $\mathbb{RP}^n$ is defined as the collection of all lines through the origin in $\mathbb{R}^{n+1}$. Other way to identify this space are as the quotient space $\mathbb{R}^{n+1} \setminus \{0\}/(x \sim \lambda x, \lambda \neq 0)$, or, equivalently, the quotient space $S^n/(x \sim -x)$. There are two claims that I want to make:
  (1) There is a CW-structure on S^n with two cells in every dimension $0 \leq k \leq n$. [We have already done this for a few spheres, and the proof for all spheres is a homework problem.]
  (2) The identifications of $x \sim -x$ in the definition of $\mathbb{RP}^n$ precisely identify the pairs of k-cells in S^n in the above presentation: if e_k, e_k' are two k-cells, then they become identified in $\mathbb{RP}^n$.

  This leads to a CW structure on $\mathbb{RP}^n$ with a single cell in each dimension.

Remark 3.12. Fun fact about real projective space! There’s a bit of set up here, so hold on.

We have seen that spheres, while seemingly simple to understand, are actually quite complicated homotopically. Other than S^1, there is not a single sphere whose homtopy groups we know entirely! We have also seen that it can be nice to construct spaces by colimits, and that there is a natural way to construct the sphere S^n such that all lower dimensional spheres S^k sit nicely inside as subcomplexes. What happens when we take the colimit of the resulting diagram?

$$\operatorname{colim}(S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots) = S^\infty.$$

This space, denoted S^∞, has a very, very surprising property: it is contractible! Thus, all of its homotopy groups vanish.

We have also seen (getting back to projective space now) that $\mathbb{RP}^n$ is a quotient of S^n by identifying antipodal points. The natural quotient maps $S^n \to \mathbb{RP}^n$ respect the identifications of the inclusions of spheres we were just talking about, meaning there is a big commutative diagram

$$\begin{array}{ccccccccc} S^0 & \longrightarrow & S^1 & \longrightarrow & S^2 & \longrightarrow & S^3 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{RP}^0 & \longrightarrow & \mathbb{RP}^1 & \longrightarrow & \mathbb{RP}^2 & \longrightarrow & \mathbb{RP}^3 & \longrightarrow & \cdots \end{array}$$

A couple things to notice: $\mathbb{RP}^0 \cong *$, and $\mathbb{RP}^1 \simeq S^1$, and $\mathbb{RP}^2$ is a gross space that can't be embedded into $\mathbb{R}^3$; for any point $x \in \mathbb{RP}^n$, the fiber over x in S^n is exactly two antipodal points.

Now, here is the cool fact! We can also take the colimit of the bottom row of the above commutative diagram, defining a space denoted $\mathbb{RP}^\infty$. This space, unlike S^∞, has exactly 1 nontrivial homotopy group! It is $\pi_1(\mathbb{RP}^\infty) \cong \mathbb{Z}/2$. Kinda wild! All of the higher groups vanish, and the order of the only nontrivial group is exactly the size of the fiber over any point. This is not a coincidence, but I will not get into it.

Be aware: CW-structures are not unique! There are easy ways to find different CW structures for all the examples above:

- For I, take instead $X_0 = \{*_0, *_1, *_2, *_3, *_4\}$ to be 5 disjoint points (thinking of them as $0, 1/4, 1/2, 3/4, 1$), and attach 4 different 1-cells to X_0. In terms of maps $f_i : S^0 \to X_0$, we have four different maps $f_1, \dots f_4$:
$$\operatorname{im}(f_1) = \{*_0, *_1\}, \quad \operatorname{im}(f_2) = \{*_1, *_2\}, \quad \operatorname{im}(f_3) = \{*_2, *_3\},$$
$$\operatorname{im}(f_4) = \{*_3, *_4\}, \quad \operatorname{im}(f_5) = \{*_4, *_5\}.$$
Then the pushout
$$\begin{array}{ccc} \coprod_{i=1}^4 S^0 & \longrightarrow & \coprod_{i=1}^4 \overline{\mathbb{D}}^1 \\ {\scriptstyle \amalg f_i}\downarrow & & \downarrow \\ X_0 & \longrightarrow & X_1 \end{array}$$
glues in line segments between each of the 0-cells in X_0, hence X_1 is again homeomorphic to I.
- Similarly, for constructing $\mathbb{R}$, we could've taken X_0 to be the points representing only the even integers and glued in 1-cells between them.

Lecture 17.

Here's a nice fact about colimits of functors $F : \mathbb{Z}_\leq \to \mathcal{T}\mathrm{op}$. Recall that these diagrams look like

$$\cdots \to X_{i-1} \to X_i \to X_{i+1} \to \cdots$$

If there is some n such that $X_n \cong X_{n+1} \cong X_{n+2} \cong \cdots$, then $\operatorname{colim} F = X_n$. So, if the diagram is eventually constant on the right, then the colimit is just that constant term. Similarly, if there is some n where $\cdots \cong X_{n-2} \cong X_{n-1} \cong X_n$, then $\lim F = X_n$. So, if the diagram is eventualy constant on the left, then the limit is just that constant term.

Consider the case where $X_n \cong X_{n+1} \cong \cdots$. The colimit of F is defined as

$$\operatorname{colim} F = \coprod_{i \in \mathbb{Z}} X_i / \sim,$$

where if $x_i \in X_i$ and $x_j \in X_j$, then $x_i \sim x_j$ is and only if there are some $r, s \geq 0$ such that

$$f_{i+r} \circ f_{i+r-1} \circ \cdots \circ f_i(x_i) = f_{j+s} \circ f_{j+s-1} \circ \cdots \circ f_j(x_j).$$

For example, if $x_{i-1} \in f_i^{-1}(x_i)$, then $x_{i-1} \sim x_i$. But this actually lets us identify everything! For all $i < n$ and any $x_i \in X_i$, we know that $f_n \circ f_{n-1} \circ \cdots f_i(x_i) \in X_n$. Thus, for any point x_i, there is some point $x_n \in X_n$ such that $x_i \sim x_n$ in $\operatorname{colim} F$. Moreover, since each map $X_{n+k} \to X_{n+k+1}$ is an isomorphism for $k \geq 0$, the relations imposed by the tail end of the diagram don't do actually do anything! That is, if we have two points $x_n \neq x'_n \in X_n$, then since each map f_{n+k} is an isomorphism, it is in particular a bijection, hence $f_{n+k}(x_n) \neq f_{n+j}(x'_n)$ for any $j, k \geq 0$, meaning $x_n \not\sim x'_n$ in $\mathrm{colim} F$. In other words, there is a homeomorphism

$$\mathrm{colim} F \to X_n.$$

CW-complexes are nice. They give us a combinatorial way to keep track of a topological space by remembering: the 0-cells, 1-cells, and so on; how these cells are attached to each other; and an algorithm for how to glue all this data together.

Here are some reasons to care about CW-complexes from the perspective of homotopy theory. These I will not prove: they're hard to prove!

Theorem 3.13 (CW-approximation)**.** *Let X be any topological space. Then there is a CW-complex Z and a continuous function*

$$f : Z \to X$$

such that $\pi_n(f) : \pi_n(Z) \to \pi_n(X)$ is an isomorphism for all $n \geq 0$.

A map $f : X \to Y$ which induces an isomorphism on homotopy groups such as above is called a *weak homotopy equivalence*. This is strictly weaker than a homotopy equivalence: there need not be a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

However, for CW-complexes, this is not the case!

Theorem 3.14. *If X, Y are CW-complexes and $f : X \to Y$ is a weak homotopy equivalence, then it is a homotopy equivalence.*

Essentially, the CW-structures of X and Y allow one to construct the homotopy inverse $g : Y \to X$ from the data of the isomorphisms $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$. What these two theorems address are the following:

Boy, wouldn't it be nice if a weak homotopy equivalence was enough to have a homotopy equivalence? Wouldn't it be nice if one could detect if two spaces are homotopy equivalent just by doing algebra?

For CW-complexes, this is true!

Boy, it's nice that that works for CW-complexes. But what about my weird tautological spaces that I like because I am a psychopath?

Well, you may not have *exactly* the same thing, but you do at least know that your weird tautological space is weakly equivalent to one of these nice CW-complexes!

For this reason, most often when people say something like "let X be a space", they really mean "let X be a CW-complex".

Lecture 18.

3.7. **Cofibrations.** A map of spaces $i : A \to X$ is a *cofibration* if for any continuous maps $h : A \times I \to Y$ and $f : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{(\mathrm{id}_A, 0)} & A \times I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

there exists some $\tilde{h} : X \times I \to Y$ making the following diagram commute:

$$\begin{array}{ccccc} A & \xrightarrow{(\mathrm{id}_A, 0)} & A \times I & \xrightarrow{i \times \mathrm{id}_I} & X \times I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle \tilde{h}} & \\ X & \xrightarrow{\ f\ } & Y & & \end{array}$$

Some people call this property the *homotopy extension property*, as we've extended the first diagram a little. You should think about cofibrations as being "good inclusions".

Another way to think about this homotopy extension condition, especially if $A \subseteq X$, is as follows. Commutativity of the first diagram implies that $h : A \times I \to Y$ is a homotopy from $H(a, 0) = f \circ i(a)$ to some other map. The space $A \times I$ naturally sits inside of a larger cylinder $X \times I$, which we can view as the pushout

$$(A \times I) \cup_{A \times \{0\}} (X \times \{0\}).$$

This space looks like the cylinder on A as a subspace inside of the cylinder on X, except that at level 0 I have expanded to all of X. We naturally have a map

$$(A \times I) \cup_{A \times \{0\}} (X \times \{0\}) \to Y,$$

which is just given by h. We can extend to all of X at level 0 because $H(a,0) = f \circ i(a)$ factors through X by construction. The homotopy extension problem can be rephrased to ask if we have a dotted line in the following diagram:

$$\begin{array}{ccc} (A \times I)_{A \times \{0\}}(X \times \{0\}) & \longrightarrow & Y \\ \downarrow & \nearrow & \\ X \times I & & \end{array}$$

In other words, since the vertical map just includes into the cylinder on X, can we extend the homotopy h to one on X?

Maybe to be a little more specific: suppose we have a map $f : V \to W$ of k-vector spaces. Something spectacular about vector spaces is how easy it is to verify conditions about linear transformations by making a clever reduction with linear algebra.

One of the first times we see this is when checking whether f is injective or not. By definition, f is injective if for all vectors $v, w \in V$, if $f(v) = f(w)$, then in fact $v = w$. This definition asks about every possible set of two vectors from V, which is a kinda ridiculous thing to check in practice. There is a clever fix coming from algebra. Suppose I only know that if $f(v) = 0$, then in fact $v = 0$. Well, consider any other two vectors $u, w \in V$, and suppose that $f(u) = f(w)$. Linear transformations commute with addition and subtraction (this is a **big deal!!!**), so we have that

$$0 = f(u) - f(w) = f(u - w).$$

But now we must have that $u - w = 0$, hence $u = w$ and f must be injective.

A cofibration is the topologists replacement for a "nice" injective map. There are a lot of more interesting continuous maps in topology, which makes it very interesting, but also (in addition to a lack of any algebra) can make it hard to work with things. Cofibrations are a more robust class of maps which are injective-like and have a condition we can check.

Remark 3.15. Another thing that's failing here: the category $\mathcal{T}op$ is not *abelian*!

Another nice fact about injective linear maps is that they are isomorphisms onto their images. There is a sort of space level analogue of this fact. For any map $f : X \to Y$ of spaces, we can construct the *mapping cylinder* of f, the topological space denoted Mf and defined by

$$Mf := (X \times I) \amalg Y)/((x,0) \sim f(x)).$$

In other words, take the cylinder on X, then glue the bottom of the cylinder onto Y at the points specified by f. There is an inclusion $i : X \to Mf$ which sends X to its copy at the top of the cylinder, and there is a projection $r : Mf \to Y$ sending Y to itself and sending any point (x, t) in the cylinder to $f(x) \in Y$.

Proposition 3.16. *For any continuous map $f : X \to Y$, the map $i : X \to Mf$ is a cofibration, and the map $r : Mf \to Y$ is a homotopy equivalence. Moreover, there is a factorization*

$$\begin{array}{ccc} & & Mf \\ & \overset{i}{\nearrow} & \downarrow r \\ X & \xrightarrow{f} & Y \end{array}$$

This proposition is saying that up to homotopy, every morphism is a cofibration. That's kinda funky!

Lecture 19. We can use cofibrations and cofibers to create topological analogues of long exact sequences. Let $f : A \to X$ be a continuous map. We have already seen that the cofiber of f, which is the pushout

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & X/A \end{array}$$

is identified with the quotient space X/A. There is a similar construction called the *cone* of f. This is the topological space $C(f)$ defined as follows. Let now A, X be based spaces, and denote the basepoint by $*$. The cone on A, denoted $C(A)$, is the space obtained by collapsing the subspace $* \times I$ in the unbased cone $C^{(u)}(A)$ to a point:

$$C(A) = C^{(u)}(A)/(\{*\} \times I)$$

This space is based. The cone on $f : A \to X$, viewed as a map of based space, is the space which glues $C(A)$ onto X as determined by the map f:

$$C(f) = C(A) \cup_A X.$$

For technical reasons, from here on out we will let our based spaces all be *well-based*, meaning that the inclusion of the basepoint $* \to X$ is a cofibration.

Remark 3.17. If $f : A \to X$ is a cofibration and both A and X are well-based, then the collapse map $C(f) \to X/A$ is a homotopy equivalence.

A *cofiber sequence* is any diagram of the form

$$A \to X \to C(f),$$

or any diagram weakly to equivalent to such a diagram. By our above remark, an example is any diagram

$$A \to X \to X/A$$

where $A \to X$ is a cofibration and A, X are well-based. How do we extend this sequence to the right? We take cofibers again! The cone on the map $X \to C(f)$ is defined as

$$C(X) \cup_X C(f).$$

This space is homotopy equivalent to what is known as the suspension ΣX, which we can define as

$$\Sigma X = C(X) \cup_X C(X)$$

This is tha magic: we can continue this process indefinitely, resulting in a sequence of cofibrations know as a *Puppe sequence*

$$A \to X \to C(f) \to \Sigma A \to \Sigma X \to \Sigma C(f) \to \Sigma^2 A \to \Sigma^2 X \to \cdots$$

Lecture 20. On the homework, we will see that for any two based space X and Y, the set of based homotopy classes of maps $[\Sigma X, Y]_*$ has a group structure. We can use this to turn the Puppe sequence, which is like an exact sequence of spaces, into an exact sequence in algebra.

Proposition 3.18. *Let Z be any based space. Applying the functor $[-, Z]_*$ to the Puppe sequence yields an exact sequence*

$$\cdots [\Sigma^2 X, Z]_* \to [\Sigma^2 A, Z]_* \to [\Sigma C(f), Z]_* \to [\Sigma X, Z]_* \to [\Sigma A, Z]_* \to [C(f), Z]_* \to [X, Z]_* \to [A, Z]_*.$$

Let's just think about exactness at $[X, Z]_*$ (technically this is only exactness of sets, but that's not too important right now). Suppose that $g \in [X, Z]_*$ such that $g \circ f = c_z \in [A, Z]_*$ is sent to the constant map at the basepoint $z \in Z$. We need to show that there is some $\varphi \in [C(f), Z]_*$ such that $\varphi \circ i \simeq g$, where $i : X \to C(f)$. In other words, we need an extension φ as in the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{f} & X & \xrightarrow{i} & C(f) = X \cup_A C(A) \\ & & {\scriptstyle g}\downarrow & \swarrow {\scriptstyle \varphi} & \\ & & Z & & \end{array}$$

Dually, we need to show that if $g \in [C(f), Z]_*$, then $f \circ i \circ g \simeq c_z$. Think about how the definition of a cofibration ensures that this must happen!

Lecture 21.

3.8. **Fibrations.** Dual to cofibrations, we say that a surjective map of spaces $p : E \to B$ is a *fibration* if for any continuous maps $f : Y \to E$ and $h : Y \times I \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{f} & E \\ {\scriptstyle (\mathrm{id}_Y,0)}\downarrow & & \downarrow{\scriptstyle p} \\ Y \times I & \xrightarrow{h} & B \end{array}$$

there exists some $\tilde{h}$ making the following diagram commute:

$$\begin{array}{ccc} Y & \xrightarrow{f} & E \\ {\scriptstyle (\mathrm{id}_Y,0)}\downarrow & \overset{\tilde{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ Y \times I & \xrightarrow{h} & B \end{array}$$

Some people call this the *homotopy extension property*, as we've lifted the map $h : Y \times I \to B$, which is a homotopy between the continuous maps $h_0, h_1 : Y \to B$, along the map p to E.

We have previously defined the fiber of a based map $p : E \to B$ as the pullback

$$\begin{array}{ccc} F_b & \xrightarrow{f} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & B \end{array}$$

If p is further a fibration, then all fibers are homotopy equivalent: if $x, y \in B$ are any two points, then $p^{-1}(x) = F_x \simeq F_y = p^{-1}(y)$. For this reason, one often just writes F for this fiber, and denotes a fibration by

$$F \to E \to B.$$

Example 3.19.

- For any spaces F, B, the projection map $\pi : F \times B \to B$ is a fibration. Each fiber is homotopy equivalent (in fact homeomorphic) to F.
- The covering map $p : \mathbb{R} \to \mathbb{S}^1$ is a fibration. Each fiber is homotopy equivalent (in fact homeomorphic) to $\mathbb{Z}$.
- We will construct a fibration
$$S^1 \to S^3 \to S^2$$
known as the Hopf fibration. Consider $S^3 \subseteq \mathbb{C}^2$ as the set of all unit vectors. Consider S^2 as the quotient of $\mathbb{C}^2 \setminus \{0\}$ where we identify $x \sim \lambda x$ for all $\lambda \in \mathbb{C} \setminus \{0\}$; in other words, we have $S^2 \simeq \mathbb{CP}^1$, the space of lines through the origin in $\mathbb{C}^2$. There is a well-defined map
$$\eta : S^3 \to S^2, \quad \eta(z_1, z_2) = [z_1/z_2],$$
or $\eta(z_1, z_2) = \infty$ for $z_2 = 0$. Notice that the fiber over any point $x \in S^2$ consists of pairs of complex numbers (z_1, z_2) such that $|z_1|^2 + |z_2|^2 = 1$ and $z_1/z_2 = x$: this exactly describes a great circle on S^3!
- What happens if we do a "real" version of the above construction? That is, there is a sphere living in $\mathbb{R}^2$, more commonly known as the S^1. There is also a quotient of $\mathbb{R}^2 \setminus \{0\}$ where identify $x \sim \lambda x$ for all $x \in \mathbb{R} \setminus \{0\}$ which gives us $\mathbb{RP}^1 \simeq S^1$. There is a well-defined map
$$t : S^1 \to S^1, \quad t(x, y) = [x/y].$$
One thing we can see is that if (x, y) and (x', y') are any two points on the circle, then $x/y = x'/y'$ if and only if $(x', y') = (x, y)$ or $(x', y') = (-x, -y)$. In particular, the fiber over any point is $t^{-1}(x, y) = \{(x, y), (-x, -y)\} \simeq S^0$! This map is a fibration, and it takes the form
$$S^0 \to S^1 \to S^1.$$
What is this map t doing? It is not hard to see: $t : S^1 \to S^1$ is the map representing $2 \in \mathbb{Z} = \pi_1(S^1)$. In other words, it wraps around the circle twice!

Lecture 22.

Similar to how we can replace any continuous map with a cofibration (up to homotopy), we can do the same for a fibration. For any map of spaces $f : X \to Y$, the *path mapping space* Nf is defined as

$$Nf = \{(x,\gamma) | x \in X, \gamma \in \mathrm{Map}(I,Y), f(x) = \gamma(0)\} \subseteq X \times \mathrm{Map}(I,Y)$$

This is the subspace of $X \times \mathrm{Map}(I,Y)$ consisting of all pairs (X,γ), where x is some point in X and $\gamma : I \to Y$ is a path starting at $\gamma(0) = f(x)$. One way to think about this space is as a subspace of the mapping space $\mathrm{Map}(I,Y)$ where we only consider those paths which start at some point $y \in Y$ which is in the image of $f : X \to Y$. We also need to be a little careful, though; in this analogy, we are counting paths with *multiplicity*, as there may be many points in X which get sent to the same point in Y.

There is a map $X \to Nf$ which sends any point $x \in X$ to the pair $(x, c_{f(x)})$ where $c_{f(x)} : I \to Y$ is the constant path at $f(x) \in Y$. There is also a map $Nf \to Y$ which sends any pair (x,γ) to the endpoint $\gamma(1) \in Y$.

Proposition 3.20. *For any continuous map $f : X \to Y$, the map $X \to Nf$ is a homotopy equivalence and the map $Nf \to Y$ is a fibration. Moreover, there is a factorization*

$$\begin{array}{ccc} & & Nf \\ & \nearrow & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Lecture 23.

Now, if with cofibrations we extended a map to the right, with fibrations we will extend to the left. Let $p : E \to B$ be any based map. The *homotopy fiber* of p is the space

$$F_p = E \times_B \mathrm{Map}_*(I,B) \times_B \{*\}$$

We should think of this as the space consisting of points $e \in E$ and paths in B from $p(e) \mapsto *$, where $* \in B$ is the basepoint. If p is already a fibration, then $F_p \simeq p^{-1}(*)$ is just the fiber over the basepoint. A *fiber sequence* is any sequence weakly homotopy equivalent to a sequence of the form

$$F_p \to E \xrightarrow{p} B.$$

Here is the key takeaway: after replacing $F_p \to E$ with a fibration (which we can do up to weak equivalence!), we can ask again what the homotopy fiber of $F_p \to E$ is. In our case, where everything is based, we have that the homotopy fiber is

$$\Omega B = \mathrm{Map}_*(S^1, B),$$

the *loop space* of B. By definition, the homotopy fiber is the space

$$F_p \times_E \mathrm{Map}_*(I,E) \times_E \{*\}.$$

So, it is the space of points of F_p and baths in E from any point $e \in E$ to the basepoint $* \in E$. Since a point of F_p itself consists of a point $e \in E$ and a path from $p(e) \mapsto *$, where $* \in B$ is the basepoint, applying p to the path in E from $e \mapsto e$ exacyly gives us a loop in B!

Rinsing and repeating, we get the fiber Puppe sequence:

$$\cdots \to \Omega^2 B \to \Omega F_p \to \Omega E \to \Omega B \to F_p \to E \to B.$$

Here is a fact dual to the one for suspensions: for any spaces X and Y, the set of based homotopy classes of maps $[X, \Omega Z]_*$ forms a group, and the set of based homotopy classes of maps $[X, \Omega^2 Z]_*$ is abelian. The group operation is concatenation. If $f, g \in [X, \Omega Z]_*$, then define $f + g$ as the map

$$X \xrightarrow{(f,g)} \Omega Z \times \Omega Z \to \Omega Z.$$

Theorem 3.21. *For any space X, applying $[X,-]_*$ to the fiber Puppe sequence gives a long exact sequence*

$$\cdots \to [X, \Omega^2 B]_* \to [X, \Omega F_p]_* \to [X, \Omega E]_* \to [X, \Omega B]_* \to [X, F_p]_* \to [X, E]_* \to [X, B]_*$$

Lecture 24. We're almost to the long exact sequence in homotopy groups! We just need one more ingredient: adjoint functors.

3.9. **Adjoints.** Suppose that $\mathcal{C}, \mathcal{D}$ are two categories. Suppose we have functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$. We say that F and G are *adjoint* if there is a bijection

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \cong \mathrm{Hom}_{\mathcal{D}}(c, G(d))$$

for all objects $c \in \mathrm{ob}(\mathcal{C})$ and $d \in \mathrm{ob}(\mathcal{D})$. In this setup, we say that F is *left-adjoint* to G, and similarly that G if *right-adjoint* to F.

Proposition 3.22. *If $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is an adjunction, then there are natural transformations*

$$\eta_{\mathcal{C}} : \mathrm{Id}_{\mathcal{C}} \implies G \circ F, \quad \varepsilon_{\mathcal{D}} : F \circ G \implies \mathrm{Id}_{\mathcal{D}}$$

known as the unit and counit of the adjunction. We sometimes also get a

Example 3.23.

- Suppose $\mathcal{C}$ is a category where the objects are "sets with some structure". We have seen that there is a forgetful functor
$$U : \mathcal{C} \to \mathcal{S}\mathrm{et}$$
which sends an object of $\mathcal{C}$ to its underlying set and morphisms to the underlying set map. Quite often, the forgetful functor admits a left adjoint known as the *free functor*, which I'll denote $F : \mathcal{S}\mathrm{et} \to \mathcal{C}$. Less often, but still sometimes, the forgetful functor will also admit a left adjoint. Sometimes this right adjoint is called the *cofree* functor. Let's look at some particular examples. In each case, try to verify that these are actually adjunctions.
  - Consider the forgetful functor
$$U : \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}.$$
This admits a left adjoint
$$F : \mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}.$$
For a set X, the topological space $F(X)$ has underlying set X with topology the discrete topology. The forgetful functor also admits a right adjoint
$$C : \mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}.$$
For a set X, the topoological space $C(X)$ has underlying set X with topology the trivial topology.
  - Consider the forgetful functor
$$U : \mathrm{Vect}_k \to \mathcal{S}\mathrm{et}.$$
This admits a left adjoint
$$F : \mathcal{S}\mathrm{et} \to \mathrm{Vect}_k$$
sending a set X to the vector space $F(X)$ with a basis element e_x for every element $x \in X$.

Proposition 3.24 (RAPL and LAPCO)**.** *Right adjoints preserve limits. Dually, left adjoints preserve colimits.*

This is really important! Lots of constructions we make in algebraic topology are by some limit or colimit construction. Knowing that some functor is a left or right adjoint means that we can "commute" this limit or colimit construction by the functor.

For example, take the forgetful functor $U : \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$. The discrete and trivial topology define left and right adjoints, respectively, meaning that the forgetful functor commutes with limits AND colimits! This is extremely useful: this implies that, if I have some diagram whose limit or colimit defines a topological space X, then the underlying set of X is the limit or colimit of the underlying sets of the diagrams! This is kind of a fact we take for granted that is extremely powerful.

Here is what is relevant for us.

Proposition 3.25. *The suspension functor Σ is left adjoint to the loop functor Ω on based homotopy classes of maps. That is, for any spaces X, Y, there is an isomorphism*

$$[X, \Omega Y]_* \cong [\Sigma X, Y]_*$$

Lecture 25. This is really an example of a more general adjunction! The category of pointed topological space $\mathcal{T}\mathrm{op}_*$ has a product which is called the *smash product*: if X, Y are pointed, then the smash product is the space

$$X \wedge Y = X \times Y / X \vee Y.$$

The smash product has some "algebra-y" properties. For example, $X \wedge S^0 \simeq X$, and $X \wedge S^1 \simeq \Sigma X$. As a consequence, $X \wedge S^n \simeq \Sigma^n X$.

Exercise 3.26. Try to visualize how $S^1 \wedge S^1 \simeq \Sigma S^1 = S^2$. By definition, $S^1 \wedge S^1 = S^1 \times S^1 / S^1 \vee S^1$. Since $S^1 \times S^1$ is the torus, this is giving us a way to construct S^2 as a quotient of the torus.

One can show that the functor $- \wedge X : \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$ is a left adjoint (there are conditions a functor must satisfy for this to be true, which one can check, but we will not discuss for lack of time). This implies that there is a right adjoint $R_X : \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$ and that the following equivalence holds for all pointed space X, Y, Z:

$$[Y \wedge X, Z]_* \cong [Y, R_X(Z)]$$

A more common name for R_X is $\mathrm{Map}_*(X, -)$! In fact, one can use this property to define the based mapping space. Since the right adjoint must exist (if these illusive conditions on checking that $X \wedge -$ is a left adjoint are checked), then there must exist some space which ought to be called $\mathrm{Map}_*(X, Z)$. In particular, this bypasses us even having to mention the compact-open topology, which is so gross!

Remark 3.27. Why "ought" R_X be called $\mathrm{Map}_*(X, -)$? Because this type of adjunction, between the product and the Hom, happens all the time. In the category of sets, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(Y \times X, Z) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(Y, \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(X, Z))$$

which is sometimes called currying. In the category of k-Vector spaces, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{V}\mathrm{ect}_k}(V \otimes U, W) \cong \mathrm{Hom}_{\mathcal{V}\mathrm{ect}_k}(V, \mathrm{Hom}_{\mathcal{V}\mathrm{ect}_k}(U, W))$$

which is known as the tensor-Hom adjunction.

Now, let $F \to E \to B$ be a fibration. Applying $[S^0, -]_*$ gives a long exact sequence

$$\cdots \to [S^0, \Omega^2 B]_* \to [S^0, \Omega F_p]_* \to [S^0, \Omega E]_* \to [S^0, \Omega B]_* \to [S^0, F_p]_* \to [S^0, E]_* \to [S^0, B]_*$$

Since Σ is left adjoint to Ω, we have that $[S^0, \Omega^n X]_* = [\Sigma^n S^0, X]_*$. But $\Sigma^n S^0 \cong S^n$, hence $[\Sigma^n S^0, X]_* \cong [S^n, X]_* \cong \pi_n(X)$. Thus, we may rewrite the above long exact sequence of homoopy groups:

$$\cdots \to \pi_2(X) \to \pi_1(F_p) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F_p) \to \pi_0(E) \to \pi_0(B).$$

Let's consider again the Hopf fibration. This took the form

$$S^1 \to S^3 \xrightarrow{\eta} S^2.$$

Thus, we have a long exact sequence

$$\cdots \to \pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1) \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) \to \pi_1(S^2) \to \cdots.$$

We have shown a few things in this class. First, we know that $\pi_1(S^1) \cong \mathbb{Z}$, and that $\pi_1(S^n) = 0$ for all $n > 1$. We kinda handwaved why $\pi_1(S^2) = 0$; this just homotopes a loop up the sphere to the basepoint. In fact, this is more general.

Proposition 3.28. *For all $k < n$, $\pi_k(S^n) = 0$.*

Another thing which I will also handwave is the following.

Proposition 3.29. $\pi_k(S^1) = 0$ *for $k > 1$.*

The proof follows from covering space theory: the covering map $p : \mathbb{R} \to S^1$ induces an isomorphism on homotopy groups π_k for $k > 1$, and $\mathbb{R} \simeq *$ and hence has no higher nontrivial homotopy groups.

There is something I wanted to get to in this class that, unfortunately, I do not have time for. This is the Freudenthal suspension theorem. Let's do a little bit of set up. Suppose that $\alpha \in \pi_k(S^n)$. Then we can represent α by a map

$$\alpha : S^k \to S^n.$$

We can apply the suspension functor $\Sigma : \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$ to this map, getting

$$\Sigma\alpha : \Sigma S^k \to \Sigma S^n.$$

But $\Sigma S^k \simeq S^{k+1}$ and $\Sigma S^n \simeq S^{n+1}$! This means that $\Sigma\alpha \in \pi_{k+1}(S^{n+1})$. All in all, this shows that suspension induces a map (which turns out to be a group homomorphism)

$$\Sigma : \pi_k(S^n) \to \pi_{k+1}(S^{n+1}).$$

I am going to slightly rewrite the indexing in the theorem for convenience.

Theorem 3.30 (Freudenthal)**.** *The suspension map*

$$\Sigma : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphism for $n > k+1$.

I will call this isomorphism region the "stable range", and denote this stable group as π_k^s.

Let's see what we can get out of the Hopf fibration long exact sequence. The two terms to the right of $\pi_1(S^1)$ are 0, so I will start writing from that point. Observe that we have then an exact sequence

$$\cdots \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to 0 \to \cdots.$$

By exactness, $\pi_2(S^2) \to \pi_1(S^1)$ must be surjective. In particular, this already shows that $\pi_2(S^2) \neq 0$! we can say more. Since $\pi_2(S^3) = 0$, this implies that our exact sequence takes the form

$$\cdots 0 \to \pi_2(S^2) \to \pi_1(S^1) \to 0 \to \cdots$$

By exactness, $\pi_2(S^2) \to \pi_1(S^1)$ is also injective, hence it is an isomorphism.

Proposition 3.31. $\pi_2(S^2) \cong \mathbb{Z}$

Now, there are suspension maps

$$\pi_1(S^1) \xrightarrow{\Sigma} \pi_2(S^2) \xrightarrow{\Sigma} \pi_3(S^3) \xrightarrow{\Sigma} \pi_4(S^4) \xrightarrow{\Sigma} \cdots$$

So, in the notation of the Freudenthal suspension theorem, $k = 0$, and the suspension map is an isomorphism as soon as $n > k+1 = 0+1 = 1$. Therefore,

Proposition 3.32. $\pi_n(S^n) \cong \mathbb{Z}$ *for all n.*

Notice that this is not true immediately from the computation of $\pi_1(S^1)$! We actually needed to compute $\pi_2(S^2)$ here, as $\pi_a(S^1)$ is not in the stable range.

Back to the Hopf fibration, we can continue to fill in terms. We have an exact sequence

$$\cdots \to \pi_4(S^2) \to \pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1) \to \pi_2(S^2) \to \cdots$$

which we can rewrite using our newfound knowledge as

$$\cdots \pi_4(S^2) \to 0 \to \mathbb{Z} \to \pi_3(S^2) \to 0 \to \mathbb{Z} \to \cdots$$

In particular, we see that $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$. This is the first homotopy group we have computed for $k > n$ which is nontrivial! In fact, the long exact sequence shows something stronger.

Proposition 3.33. *For $k \geq 3$, $\pi_k(S^3) \cong \pi_k(S^2)$.*

In the long exact sequence, the two homotopy groups above are sandwiched between two 0's as the higher homotopy of S^1 is trivial, and hence we get the isomorphism. So, even though we don't know what these higher homotopy groups are, we at least no that once we have one for S^2, we also have it for S^3!

I'll remark a few things.

- The group $\pi_3(S^2) \cong \mathbb{Z}$ is generated by the Hopf map $\eta : S^3 \to S^2$. However, this is *not* yet in the stable range! The stable stem π_1^s is given by $\pi_4(S^3)$. The suspension map
$$\Sigma : \pi_3(S^2) \to \pi_4(S^3)$$
it turns out is surjective, and so it suffices to determine the order of $\Sigma\eta \in \pi_4(S^3)$. **SURPRISE!** IT turns out that $\Sigma\eta \cdot \Sigma\eta = 0 \in \pi_4(S^3)$, hence $\pi_4(S^3) \cong \mathbb{Z}/2$.
- The other Hopf maps $\nu : S^7 \to S^4$ and $\sigma : S^{15} \to S^8$ are also nontrivial. The groups $\pi_7(S^4)$ and $\pi_{15}(S^8)$ are therefore nontrivial. Determining what they are is quite complicated: it turns out that both of these group consists of a $\mathbb{Z}$ and a torsion summand, and that the Hopf map is the generator for the $\mathbb{Z}$ summand.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON
Email address: jackmann@uw.edu