

MATH 480: HOMOTOPY THEORY

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CONTENTS

1. Introduction

2

1. INTRODUCTION

Lecture 1

The things that mathematicians care about can often be distinguished by two qualities: *existence* and *uniqueness*. That is, suppose a mathematician thinks of some concept X . There are two obvious questions to ask:

- Does such a concept X actually exist? If so, can we explicitly construct it?
- If a concept X exists, is it unique? If it isn't unique, can we classify all such types of concepts X ?

In this course, we will be studying topological spaces. Recall that a *topological space* is a set X together with a collection \mathcal{T} of subsets, called the *topology*, such that:

- $\emptyset, X \in \mathcal{T}$,
- for any collection $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ we have that $\bigcup_{i \in I} U_i \in \mathcal{T}$, and
- for any finite collection $\{U_i\}_{i=1}^n$ we have that $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We will refer to the elements of \mathcal{T} as *open sets*. It is not hard to see that topological spaces exist.

Example 1.1.

- If X is any metric space, then one can place a topology on X by letting

$$\mathcal{T} = \{B_\varepsilon(x) : x \in X, \varepsilon > 0\}.$$

In particular, \mathbb{R}^n and \mathbb{C}^n are topological spaces under the usual metric.

- If Y is a subset of any topological space X , then Y inherits a subspace topology, where the open sets in Y are just the open sets in X which intersect Y . In particular, the *n-sphere*

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} \subseteq \mathbb{R}^{n+1}$$

is a topological space.

- If X is any set, then there are always two ways to topologize X : the *discrete topology* has open sets every possible subset of X , and the *trivial topology* has open sets just \emptyset and X .

Well, since topological spaces exist, and we can explicitly construct them, let's move on to the second point. Can we *classify* topological spaces?

There is a strict way to attempt to approach this question: classify spaces up to *homeomorphism*. Recall that a *continuous function* $f : X \rightarrow Y$ is a function such that the preimage $f^{-1}(U) \subseteq X$ of any open set $U \subseteq Y$ is open, and a *homeomorphism* is a continuous function such that there exists another continuous function $g : Y \rightarrow X$, which we'll call the *inverse*, such that

$$g \circ f = \text{id}_X : X \rightarrow X, \quad f \circ g = \text{id}_Y : Y \rightarrow Y.$$

⚠ Warning ⚠ 1.2. It is not enough for a continuous function to be a bijection to be a homeomorphism.

Exercise 1.3. Construct a continuous bijection which is not a homeomorphism.

It is not an easy task to determine whether or not two random spaces are homeomorphic. If we want to be really general and consider all topological spaces (or at least most of them), then classifying them *up to homeomorphism* seems a little out of reach.

There are two clear directions to move forward from this obstacle. On the one hand, we can put more structure on our spaces and impose a stricter notion of equivalence. For example, we could only consider *manifolds*, which are topological spaces that are locally modeled by \mathbb{R}^n , or we could consider *differentiable manifolds*, which have structures amenable to the tools of calculus, or we could consider *varieties*, which are the zero-loci of systems of polynomial equations. Each of these types of spaces has a more fine-tuned notion of equivalence which allows us to better classify that type of space.

This direction will not be the focus of this course. Instead, we will *loosen* our notion of equivalence and study spaces up to *homotopy equivalence*. If you didn't think that topology was flexible enough, what with the open interval being homeomorphic to the real number line, well then get ready for some real gymnastics.

Let $f, g : X \rightarrow Y$ be continuous maps. A *homotopy* between f and g is a continuous function

$$H : X \times I \rightarrow Y$$

where $I = [0, 1]$ denotes the unit interval (which we think of as a time parameter), such that at time $t = 0$, we have $H(x, 0) = f(x)$, and at time $t = 1$, we have $H(x, 1) = g(x)$. We will often use the notation $f \simeq g$ if there is a homotopy from f to g and say that they are *homotopic*.

Example 1.4. Define two continuous maps $f, g : \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$f(x) = (x, x^2), \quad g(x) = (x, x).$$

We can define a homotopy between f and g quite easily: let $H : \mathbb{R} \times I \rightarrow \mathbb{R}^2$ be defined by

$$H(x, t) = (x, x^2 - tx^2 + tx).$$

Then $H(x, 0) = (x, x^2) = f(x)$ and $H(x, 1) = (x, x) = g(x)$. This can be seen graphically here <https://www.desmos.com/calculator/acoupulhy1>.

Exercise 1.5. Show that the notion of homotopy defines an equivalence relation on the set of continuous functions from X to Y by showing the following:

- If $f : X \rightarrow Y$ is any function, then there is a homotopy $H : X \times I \rightarrow Y$ from f to itself.
- If $H : X \times I \rightarrow Y$ is a homotopy from f to g , then there is a homotopy $H' : X \times I \rightarrow Y$ from g to f .
- If $f, g, h : X \rightarrow Y$ are continuous functions such that $f \simeq g$ and $g \simeq h$, then there is a homotopy $H : X \times I \rightarrow Y$ from f to h .

A *homotopy equivalence* is a continuous function $f : X \rightarrow Y$ such that there exists another continuous function $g : Y \rightarrow X$ such that

$$g \circ f \simeq \text{id}_X : X \rightarrow X, \quad f \circ g \simeq \text{id}_Y : Y \rightarrow Y.$$

We will often write that $X \simeq Y$ if X is homotopy equivalent to Y .

Example 1.6. Consider the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}^2$ from the previous example. Essentially the same homotopy between the functions f and g shows that the subspaces

$$\text{im}(f), \text{im}(g) \subseteq \mathbb{R}^2,$$

i.e. the graphs of the functions, are homotopy equivalent.

Notice that this is very similar to the notion of homeomorphism; we have just required that the compositions be *homotopic* to the identity instead of equal. And indeed, this is a weaker notion.

Proposition 1.7. Let $f : X \rightarrow Y$ be a homeomorphism. Then f is a homotopy equivalence.

Proof. Since f is a homeomorphism, there is a continuous function $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. Define a function $H : X \times I \rightarrow X$ by $H(x, t) = (g \circ f)(x)$. This is continuous: if $U \subseteq X$ is open, then

$$H^{-1}(U) = (g \circ f)^{-1}(U) \times I = (\text{id}_X)^{-1}(U) \times I = U \times I.$$

This is open in the product topology on $X \times I$. Moreover, $H(x, 0) = (g \circ f)(x)$, and $H(x, 1) = (g \circ f)(x) = \text{id}_X(x) = x$, so H is a homotopy between $g \circ f$ and id_X . The same argument works to show that $f \circ g \simeq \text{id}_Y$. \square

To illustrate how loose of a notion homotopy equivalence is, consider the following example.

Example 1.8. Let $\overline{\mathbb{D}^n}$ be the closed unit disc in \mathbb{R}^n , regardless of n , and let $*$ denote, well, a point. It is not hard to see that $\overline{\mathbb{D}^n}$ is homotopy equivalent to $*$: let $f : \overline{\mathbb{D}^n} \rightarrow *$ be the only map possible (which sends every point in the disc to the point) and let $g : * \rightarrow \overline{\mathbb{D}^n}$ send $*$ to 0 (or honestly to any point in the disc). Then we have

$$f \circ g : * \rightarrow *, \quad (f \circ g)(*) = *,$$

and

$$g \circ f : \overline{\mathbb{D}^n} \rightarrow \overline{\mathbb{D}^n}, \quad (g \circ f)(x) = g(*) = 0.$$

It is worth checking (and not hard to show) that

$$H : * \times I \rightarrow *, \quad H(*, t) = *,$$

and

$$H' : \overline{\mathbb{D}^n} \times I \rightarrow \overline{\mathbb{D}^n}, \quad H(x, t) = t \cdot x$$

give the desired homotopy equivalences $g \circ f \simeq \text{id}_{\overline{\mathbb{D}^n}}$ and $f \circ g \simeq \text{id}_*$.

Something the above example shows is: if you had any preconceived notion of dimension, just know that homotopy equivalences couldn't care less! We will call a space which is homotopy equivalent to a point *contractible*. Notice that the above argument also shows that \mathbb{R}^n itself is contractible.

⚠ Warning ⚠ 1.9. It is **NOT TRUE** that if X is contractible and $Y \subseteq X$ is a subspace, then Y is contractible.

It turns out that, except in some particularly cherry-picked examples like the one above, it is still not very easy to determine if two randomly selected spaces are homotopy equivalent. There are a lot of ways to try and address this problem; one of the ways we will address it is by introducing *invariants* known as *homotopy groups*.

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