

## MATH 480: HOMOTOPY THEORY HOMEWORK 6

ABSTRACT. Homework 6 due in class on **Friday, May 15**.

Let  $I : A \rightarrow X$  be a continuous map of spaces. We say that  $A$  is a *retract* of  $X$  if there is a continuous map  $r : X \rightarrow A$ , called a *retraction*, such that  $r \circ i = \text{id}_A$ .

Let  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  be a short exact sequence of abelian groups. By this, we mean that the image of any map is equal to the kernel of the following map:

- $\text{im}(0 : 0 \rightarrow A) = \ker(f)$
- $\text{im}(f) = \ker(g)$
- $\text{im}(g) = \ker(0 : C \rightarrow 0)$

We say that the short exact sequence *splits* if there is an isomorphism  $B \cong A \oplus C$ .

### 1. PROBLEMS

- (1) Let  $i : A \rightarrow X$  be a cofibration. Show that there is closed set  $C \subseteq X$  which contains the image  $i(A)$  such that  $i(A)$  is a retract of  $C$ . [**Hint:** Think about the homotopy extension property and the space  $(A \times I) \cup_{A \times \{0\}} (X \times \{0\})$ , and consider the preimage of  $A \times [1/2, 1]$  intersected with  $X \times \{1\}$ .]
- (2) Show that if  $0 \rightarrow V \rightarrow W \rightarrow U \rightarrow 0$  is a short exact sequence of vector spaces, then it must always split.
- (3) Give an example of a short exact sequence of abelian groups which does not split.
- (4) Let  $X, Y$  be based topological spaces, and let  $[X, Y]_*$  denote the set of based homotopy classes of maps from  $X$  to  $Y$ . Show that
  - (a) The set  $[\Sigma X, Y]_*$  has a group structure, and that
  - (b) The set  $[\Sigma^2 X, Y]_*$  has an abelian group structure.**[Hint:** Note that the elements of  $[\Sigma X, Y]_*$  are homotopy classes of maps  $f : \Sigma X \rightarrow Y$ , and there is a map  $\Sigma X \rightarrow \Sigma X \wedge \Sigma X$  which pinches the suspension at the equator.]