

## MATH 480: HOMOTOPY THEORY HOMEWORK 5

ABSTRACT. Homework 5 due in class on **Friday, May 8**.

Let  $G$  be a group, and recall the category  $BG$  with one object  $*$  and where  $\text{Hom}_{BG}(*, *) = G$ .

### 1. PROBLEMS

- (1) A functor  $F : BG \rightarrow \text{Set}$  is the data of a set  $X = F(*)$  equipped with set functions  $g : X \rightarrow X$  for every element  $g \in G$ . This is a fancy way of saying that we have given  $X$  a group action by  $G$ .

There are two constructions related to this group that we can consider: the *fixed points*

$$X^G = \{x \in X \mid g \cdot x = x \forall g \in G\}$$

and the *orbits*

$$X/G = \coprod_{x \in X} \{g \cdot x \mid g \in G\}.$$

Prove that  $\text{colim} F = X/G$  and  $\text{lim} F = X^G$ .

- (2) Let  $K : B\mathbb{Z}/2 \rightarrow \text{Set}$  be the functor which sends the unique object in  $B\mathbb{Z}/2$  to the set  $X = \{a, b, c, d\}$  and where  $K(g) : X \rightarrow X$  is given by

$$a \mapsto b, b \mapsto a, c \mapsto d, d \mapsto c$$

for  $g \in \text{Hom}_{B\mathbb{Z}/2}(*, *)$  the non-identity element.

Let  $M : B\mathbb{Z}/4 \rightarrow \text{Set}$  be the functor which sends the unique object in  $B\mathbb{Z}/4$  to the same set  $X = \{a, b, c, d\}$ , and where if  $\gamma \in \text{Hom}_{B\mathbb{Z}/4}(*, *)$  represents a generator of the group  $\mathbb{Z}/4$ , then  $M(\gamma) : X \rightarrow X$  is given by

$$a \mapsto b, b \mapsto c, c \mapsto d, d \mapsto a.$$

Compute and compare  $\text{colim} K$ ,  $\text{lim} K$ ,  $\text{colim} M$ , and  $\text{lim} M$ . (**Hint:** It may be useful to think about the set  $X$  as the corners of a square. What do the maps  $K(g)$  and  $M(\gamma)$  do to the square?)

- (3) For each of the following spaces  $X$ , give **two different CW structures**. By different, I mean that if  $X'$  and  $X''$  are CW-complexes which are homeomorphic to  $X$ , then  $X'_k \neq X''_k$  for some  $k$ . That is,  $X'$  and  $X''$  aren't built out of the exact same set of cells.
- (a)  $X = S^1$
  - (b)  $X = S^2$
  - (c)  $X = S^n$
  - (d)  $X = \mathbb{T} = S^1 \times S^1$  (the torus)
  - (e)  $X = M$  (the mobius strip)