MATH 480: HOMOTOPY THEORY HOMEWORK 4

ABSTRACT. Homework 3 due in class on **Friday, May 1**.

1. PROBLEMS

(1) Show that the forgetful functor $U : \mathcal{V}\mathrm{ect}_k \to \mathcal{S}\mathrm{et}$ from k-vector spaces to $\mathcal{S}\mathrm{et}$ is representable, and that its representing object is the one dimensional vector space k (viewed as a vector space over itself).

(2) Show that the forgetful functor $U : \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ is representable and determine its representing object.

(3) Let $\mathcal{C}$ be the category with 3 objects $\{x, y, z\}$, unique morphisms $x \to z$ and $y \to z$, and no morphisms from x to y; in other words, $\mathcal{C}$ looks like

$$\begin{array}{ccc} & & x \\ & & \downarrow \\ y & \longrightarrow & z \end{array}$$

Consider the functor $F : \mathcal{C} \to \mathcal{T}\mathrm{op}$ where $F(x) = \mathbb{R}, F(y) = *$, and $F(z) = S^1$, and where the unique map $y \to z$ is sent to the map $* \xrightarrow{1} \mathbb{R}$ and the unique map $x \to z$ is sent to the covering map $p : \mathbb{R} \to S^1$, i.e. $p(t) = e^{2\pi i \cdot t}$; in other words, F creates the following diagram in $\mathcal{T}\mathrm{op}$:

$$\begin{array}{ccc} & & \mathbb{R} \\ & & \downarrow p \\ * & \xrightarrow{1} & S^1 \end{array}$$

Compute the limit $\lim F$, i.e. the pullback of the diagram in $\mathcal{T}\mathrm{op}$.

(4) Let $\mathcal{D}$ be the category with 3 objects $\{a, b, c\}$, unique morphisms $a \to b$ and $a \to c$, and no morphisms from b to c; in other words, $\mathcal{D}$ looks like

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \\ c & & \end{array}$$

Consider the functor $G : \mathcal{D} \to \mathcal{T}\mathrm{op}$ where $G(a) = S^1$, $G(b) = S^1 \vee S^1$, and $G(c) = \overline{\mathbb{D}}^2$, and where the unique map $a \to b$ is sent to the map which goes once around the left hand circle, then once around the right hand circle, then once around the left hand circle in the opposite direction, then once around the right hand circle in the opposite direction, and where the unique map $a \to c$ is sent to the inclusion map $S^1 \to \overline{\mathbb{D}}^2$ along the boundary; in other words, G creates the following diagram in $\mathcal{T}\mathrm{op}$:

$$\begin{array}{ccc} S^1 & \longrightarrow & S^1 \vee S^1 \\ \downarrow & & \\ \overline{\mathbb{D}}^2 & & \end{array}$$

Come the colimit $\mathrm{colim} F$, i.e. the pushout of the diagram in $\mathcal{T}\mathrm{op}$.