MATH 480: HOMOTOPY THEORY HOMEWORK 3 SOLUTIONS

ABSTRACT. Homework 3 solutions.

PROBLEM 1

Recall that if X is a topological space, then we can associate to it a category $\Pi_1(X)$ called the fundamental groupoid of X. The objects of $\Pi_1(X)$ are the points of X, and a morphism $f \in \mathrm{Hom}_{\Pi_1(X)}(x, y)$ is given by a path-homotopy class of a path $f : I \to X$ from x to y.

(a) Show that the set of endomorphisms

$$\mathrm{End}_{\Pi_1(X)}(x) := \mathrm{Hom}_{\Pi_1(X)}(x, x)$$

of any object $x \in \mathrm{ob}(\Pi_1(X))$ forms a group under function composition. This is called the group of automorphisms of x.

(b) Show that there is an isomorphism of groups

$$\mathrm{End}_{\Pi_1(X)}(x) \cong \pi_1(X, x)$$

for any $x \in \mathrm{ob}(\Pi_1(X))$.

ANSWER:

For (a), there is not much to do. Notice that morphism composition in $\Pi_1(X)$ is the same as path-concatenation, so all of the properties we want to show to make $\mathrm{End}_{\Pi_1(X)}(x)$ a group we have already shown! Thus, for any $x \in \mathrm{ob}(\Pi_1(X))$,

- The constant path $c_x : I \to X$ serves as the identity of the group, as $c_x \cdot f \simeq f \cdot c_x \simeq f$ for any loop $f : I \to X$ based at x (i.e. any morphism $f \in \mathrm{Hom}_{\Pi_1(X)}(x, x)$).
- If $f \in \mathrm{Hom}_{\Pi_1(X)}$, then there is the reverse loop $f^{\mathrm{rev}} \in \mathrm{Hom}_{\Pi_1(X)}(x, x)$ such that $f \cdot f^{\mathrm{rev}} \simeq f^{\mathrm{rev}} \cdot f \simeq c_x$, giving inverses.
- Finally, if $f, g, h \in \mathrm{Hom}_{\Pi_1(X)}(x, x)$, then $f \cdot (g \cdot h) \simeq (f \cdot g) \cdot h$, giving associativity.

For (b), by definition the set on which $\pi_1(X, x)$ is defined is the path homotopy classes of loops based at x. This is the same as the set $\mathrm{Hom}_{\Pi_1(X)}(x, x)$. The group operation is defined in the same way, so these groups are isomorphic.

(**NOTE:** In fact, if X is an object in *any* category $\mathcal{C}$, one always group of automorphisms $\mathrm{Aut}_{\mathcal{C}}(X)$ of *invertible* morphisms $f : X \to X$. The neat thing about $\Pi_1(X)$, or any groupoid for that matter, is that *every* morphism in sight is invertible, hence $\mathrm{Aut}_{\Pi_1(X)}(x) \cong \mathrm{End}_{\Pi_1(X)}(x)$. In general, this need not me an isomorphism, as not all morphisms in a category are invertible (not every endomorphism of a set is a bijection, not every endomorphism of a group is a group isomorphism, etc).)

PROBLEM 2

In this problem, we will interpret the determinant as a natural transformation. Let $f: k \to k'$ be a field homomorphism.

(1) Show that there is a functor
$$\mathrm{GL}_n : \mathcal{F}\mathrm{ield} \to \mathcal{G}\mathrm{rp}$$
which sends a field k to the group $\mathrm{GL}_n(k)$ of invertible $n \times n$-matrices with entries in k. (**Note:** Why is $\mathrm{GL}_n(k)$ a group? Is it necessarily abelian?)

(2) Show that there is a functor
$$(-)^* : \mathcal{F}\mathrm{ield} \to \mathcal{G}\mathrm{rp}$$
which sends a field k to the group k^* of invertible elements of k under multiplication. (**Note:** Can we interpret k^* in the language of part (a)?)

(3) Taking for granted that the determinant of an $n \times n$-matrix can be upgraded to a group homomorphism $\det : \mathrm{GL}_n(k) \to k^*$, show that the determinant can be further upgraded to a natural transformation
$$\det : \mathrm{GL}_n \implies (-)^*$$

ANSWER:

For (a), we want to check that the operation
$$\mathrm{GL}_n : \mathcal{F}\mathrm{ields} \to \mathcal{G}\mathrm{rp}$$
which on objects sends a field k to the set $\mathrm{GL}_n(k)$ of invertible $n \times n$-matrices and on morphisms sends $f: k \to k'$ sends to the map $\mathrm{GL}_n(f) : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k')$ which takes an invertible matrix $A = (a_{ij}) \in \mathrm{GL}_n(k)$ to the matrix $(f(a_{ij}))$ defines a functor. Let's do it!

First, observe that under matrix multiplication, the set $\mathrm{GL}_n(k)$ is indeed a group. The identity matrix I_n serves as the identity, every matrix A in $\mathrm{GL}_n(k)$ is invertible, hence has an inverse matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$, and matrix multiplication is associative. (**NOTE:** Matrix multiplication is not commutative, so $\mathrm{GL}_n(k)$ is not in general abelian!)

Additionally, if $f: k \to k'$ is a field homomorphism, then the induced map $\mathrm{GL}_n(f) : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k')$ is indeed a group homomorphism. If I_n denotes the identity matrix over k, then our formula says that $\mathrm{GL}_n(f)$ sends I_n to the matrix over k' whose entries are given by applying f to the entries of I_n. Since f is a field homomorphism, it must send $0 \mapsto 0$ and $1 \mapsto 1$, hence $I_n \mapsto I_n$. Moreover, if $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices over k, then $AB = (\sum a_{ik}b_{kj})$ and so $\mathrm{GL}_n(f)$ sends AB to the matrix $(f(\sum a_{ik}b_{kj}))$. Since f is a map f fields, it commutes with addition and multiplication, thus
$$f(\sum a_{ik}b_{kj}) = \sum f(a_{ik})f(b_{kj}).$$
These are precisely the entries of the matrix we get by applying $\mathrm{GL}_n(f)$ to A, then to B, then multiplying in $\mathrm{GL}_n(k')$. In other words, $\mathrm{GL}_n(f)$ is a group homomorphism.

Finally, we show that GL_n is a functor (everything above was kinda just setup!). The identity map $\mathrm{id}_k : k \to k$ gets mapped to the group homomorphism $\mathrm{GL}_n(\mathrm{id}_k) : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$. On matrices, this sends a matrix A to the matrix where I've applied the identity function to each entry of A. In other words, $\mathrm{GL}_n(\mathrm{id}_k) = \mathrm{id}_{\mathrm{GL}_n(k)}$. Finally, if $f: k \to k'$ and $g: k' \to k''$, then $\mathrm{GL}_n(g \circ f) : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k'')$ sends a matrix A to the matrix obtained by applying $g \circ f$ to the entries of A, while the composition
$$\mathrm{GL}_n(k) \xrightarrow{\mathrm{GL}_n(f)} \mathrm{GL}_n(k') \xrightarrow{\mathrm{GL}_n(g)} \mathrm{GL}_n(k'')$$
first takes the matrix A, applies g to its entries, then applies f to those entries. These are describing the exact same process. Thus, GL_n is a functor.

For (b), I will be a little more brief and cheat: for any field k, there is an isomorphism $\mathrm{GL}_1(k) \cong k^*$! An invertible 1×1-matrix is determined by exactly one invertible element of k. Thus, we already proved that $(-)^*$ was a functor in part (a).

For (c), we need to show that if $f : k \to k'$ is any field homomorphism, then we have the following commutative diagram of groups:

$$\begin{array}{ccc} \mathrm{GL}_n(k) & \xrightarrow{\det_k} & k^* \\ {\scriptstyle \mathrm{GL}_n(f)}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ \mathrm{GL}_n(k') & \xrightarrow[\det_{k'}]{} & k'^* \end{array}$$

So, let $A = (a_{ij}) \in \mathrm{GL}_n(k)$. We first go across the top of the diagram, then down. The determinant is defined as a product over all permutations in the following way (there are other ways to define the determinant which do not use the symmetric group):

$$\det_k(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)},$$

where $\mathrm{sgn}(\sigma) \in \{1, -1\}$ is the sign of the permutation σ. Then, the map $f^* : k^* \to k'^*$ just applies f to this sum. Since this map is induced by a field homomorphism, it commutes with multiplication and addition, and therefore

$$f^*(\det_k(A)) = f^*(\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) f(a_{1,\sigma(1)}) f(a_{2,\sigma(2)}) \cdots f(a_{n,\sigma(n)}).$$

Now, let's take $A \in \mathrm{GL}_n(k)$ and traverse the other way around the diagram. We know that $\mathrm{GL}_n(f)$ takes $A = (a_{ij})$ to the matrix $(f(a_{ij})) \in \mathrm{GL}_n(k')$. Now, we get to compute the determinant again! We arrive at

$$\det_{k'}(f(a_{ij})) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) f(a_{1,\sigma(1)}) f(a_{2,\sigma(2)}) \cdots f(a_{n,\sigma(n)}).$$

Lookie here lookie here, this is the same formula as we got going the other way around! Thus, we have a natural transformation

$$\det : \mathrm{GL}_n(-) \implies (-)^*.$$