MATH 480: HOMOTOPY THEORY HOMEWORK 3

ABSTRACT. Homework 3 due in class on **Wednesday, April 22**.

1. PROBLEMS

(1) Recall that if X is a topological space, then we can associate to it a category $\Pi_1(X)$ called the fundamental groupoid of X. The objects of $\Pi_1(X)$ are the points of X, and a morphism $f \in \mathrm{Hom}_{\Pi_1(X)}(x, y)$ is given by a path-homotopy class of a path $f : I \to X$ from x to y.

(a) Show that the set of endomorphisms

$$\mathrm{End}_{\Pi_1(X)}(x) := \mathrm{Hom}_{\Pi_1(X)}(x, x)$$

of any object $x \in \mathrm{ob}(\Pi_1(X))$ forms a group under function composition. This is called the group of automorphisms of x.

(b) Show that there is an isomorphism of groups

$$\mathrm{End}_{\Pi_1(X)}(x) \cong \pi_1(X, x)$$

for any $x \in \mathrm{ob}(\Pi_1(X))$.

(2) In this problem, we will interpret the determinant as a natural transformation. Let $f : k \to k'$ be a field homomorphism.

(a) Show that there is a functor

$$\mathrm{GL}_n : \mathcal{F}\mathrm{ield} \to \mathcal{G}\mathrm{rp}$$

which sends a field k to the group $\mathrm{GL}_n(k)$ of invertible $n \times n$-matrices with entries in k. (**Note:** Why is $\mathrm{GL}_n(k)$ a group? Is it necessarily abelian?)

(b) Show that there is a functor

$$(-)^* : \mathcal{F}\mathrm{ield} \to \mathcal{G}\mathrm{rp}$$

which sends a field k to the group k^* of invertible elements of k under multiplication. (**Note:** Can we interpret k^* in the language of part (a)?)

(c) Taking for granted that the determinant of an $n \times n$-matrix can be upgraded to a group homomorphism $\det : \mathrm{GL}_n(k) \to k^*$, show that the determinant can be further upgraded to a natural transformation

$$\det : \mathrm{GL}_n \implies (-)^*$$