MATH 480: HOMOTOPY THEORY HOMEWORK 2 SOLUTIONS

ABSTRACT. Homework 2 solutions.

PROBLEM 1 (FUNDAMENTAL THEOREM OF ALGEBRA)

Let $f(x)$ be the complex polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where a_i are not all zero. Then there exists some complex number $z \in \mathbb{C}$ such that $f(z) = 0$. (**Hint:** for sake of contradiction, suppose that $f(x) \neq 0$ for any $x \in \mathbb{C}$. Viewing $S^1 \subseteq \mathbb{C}$, this allows one to define a continuous function $\hat{f} : S^1 \to S^1$ by

$$\hat{f}(x) = \frac{f(x)}{||f(x)||}.$$

Use $\hat{f}$ to contradict $\pi_1(S^1) \cong \mathbb{Z}$.)

ANSWER:

We'll proceed by following the hint. We may as well assume that $f(x)$ is monic by scaling by $\frac{1}{a_n}$. Consider the function $\hat{f} : I \times \mathbb{R} \to S^1$ defined by

$$\hat{f}(s,r) = \frac{f(r \cdot e^{2\pi i \cdot s})/f(r)}{||f(r \cdot e^{2\pi i \cdot s})/f(r)||.}$$

This is continuous as polynomials and scaling are continuous functions. Moreover, this is well-defined since $f(x) \neq 0$ for all $x \in \mathbb{C}$. Additionally, we see that $g(0,r) = g(1,r)$, hence the universal property of the quotient along $\mathbb{R} \to S^1$ implies that $\hat{f}$ descends to a map

$$\hat{f} : S^1 \times \mathbb{R} \to S^1.$$

Notice as well that we can use this parameter space $\mathbb{R}$ to construct a homotopy of loops! Meaning, for any $r \in \mathbb{R}$, there is a continuous function

$$H_r : S^1 \times I \to S^1$$

given by $H_r(t,s) = \hat{f}(t \cdot r, s)$. Since $\hat{f}(s,0) = 1$, H_r is a homotopy between $\hat{f}(r,-) : S^1 \to S^1$ and the constant path at 1. Since this is the identity element of $\pi_1(S^1)$, this implies that in fact all of the loops $\hat{f}(r,-)$ are nullhomotopic.

Fix some point $x_0 \in \mathbb{C}$ such that $p = |x_0| > \sum_{i=1}^{n-1} |a_i|$ and $p > 1$, and consider the circle C in $\mathbb{C}$ centered at 0 of radius p. Notice that this circle, by construction, contains all of the coefficients of f. Using this, we have for all x on C that

$$|x^n| = p^n > (\sum_{i=1}^{n-1} |a_i|) \cdot |x^{n-1}| \geq |a_{n-1}x^{n-1} + \cdots + a_1 x + a_0|.$$

Further, the map $g : C \times I \to \mathbb{C}$ given by $g(x,t) = x^n + t \cdot (a_{n-1}x^{n-1} + \cdots a_1 x + a_0)$ is never 0. In other words, it passes to a map $g : C \times I \to \mathbb{C} - \{0\}$. Notice that there are homotopy equivalences $C \cong \mathbb{C} - \{0\} \cong S^1$.

Finally, we define our homotopy which will bring about a contradiction. Let $H : \mathbb{R} \times I \to S^1$ be defined by

$$H(r,t) = \frac{g(pe^{2\pi i \cdot s}, t)/g(p,t)}{||g(pe^{2\pi i \cdot s}, t)/g(p,t)||}$$

At time 1, this is just the map which traverses around C n times. At time 0, this is $f(z_0, -)$, which we have shown is nullhomotopic. Since ω_n is not the trivial element of $\pi_1(S^1)$, we have our contradiction.

PROBLEM 2

Let X be some space which is the unions of some path connected open subsets $\{U_\alpha\}$, each of which containing the basepoint $x_0 \in X$ (i.e. $x_0 \in U_\alpha$ for all α), and such that each pairwise intersection $U_{\alpha_1} \cap U_{\alpha_2}$ is path-connected. Prove that if $\gamma : I \to X$ is any loop in X based at x_0, then there are loops $\gamma_\alpha : I \to U_\alpha$ based at x_0 for every U_α such that

$$\gamma \simeq \gamma_{\alpha_1} \cdot \gamma_{\alpha_2} \cdot \gamma_{\alpha_2} \cdot \cdots$$

In other words, any element $\gamma \in \pi_1(X, x_0)$ can be rewritten as a product of loops $\gamma_\alpha \in \pi_1(U_\alpha, x_0)$ for each U_α. As a consequence, show that $\pi_1(S^n) = 0$ for $n \geq 2$.

ANSWER:

Take our loop $\gamma : I \to X$ at the basepoint x_0. The point is that since the intersections of these open sets U_α are all open, we can divvy up the interval I into subintervals

$$I = [0, t_1] \cup [t_1, t_2] \cup [t_2, t_3] \cup \cdots \cup [t_{n-1}, 1]$$

where $t_i \neq t_j$ such that the segment on the loop $\gamma([t_k, t_{k+1}])$ lies completely in some U_α. It's not even that hard to do it! If $x \in I$ is any point in the interval, then continuity and openness of U_α imply that there is some open neighborhood V_x of x in I (a small little open interval about x) which gets mapped entirely into U_α. Then the open sets $\{V_x\}_{x \in I}$ form an open cover of I. Since I is compact, these is a finite subcollection of these open intervals which must cover I: this defines our desired subintervals. (Notice that the open sets U_α overlap nontrivially by assumption, so there's no "messy transition" over the boundary of one open set to another in this divvying up business.)

For any subinterval $[t_k, t_{k+1}]$, let $\gamma_k : I \to X$ denote the path (entirely contained in some U_α!) where $\gamma_k(0) = \gamma(t_k)$ and $\gamma(t_{k+1})$. By construction, we have that $\gamma = \gamma_{n-1} \cdots \gamma_0$. Now, we just use path connectedness to construct paths $\varphi_k : I \to X$ from $\gamma_k(1) = \gamma(t_k)$ to x_0 completely contained in whatever U_α the image of γ_k is contained in. Then, we have that

$$\gamma = (\gamma_{n-1} \cdot \varphi_{n-1}) \cdot (\varphi_{n-1}^{\text{rev}} \cdot \gamma_{n-2} \cdot \gamma_{n-2}) \cdots (\gamma_0^{\text{rev}} \cdot \varphi_0).$$

Each term in parentheses is a loop based at x_0 contained entirely in one of the U_α, finishing the proof.

To show that $\pi_1(S^n) = 0$ for $n \geq 2$, just note that we can cover S^n by two open subsets N, S, one of which is the northern hemisphere and a smidge over the equator and one of which is the southern hemisphere and a smidge over the equator. Then $N \cap S \cong S^{n-1} \times (0,1)$ (think about this in the case of $n = 2$: the intersection is a thick open belt around the equator). Notice as well that both N and S are homeomorphic to $\mathbb{R}^n$. If $\gamma : I \to S^n$ is any loop based at some basepoint x in $N \cap S$, then we can decompose it into into $\gamma_N \cdot \gamma_S$, where γ_N is contained in N and γ_S is contained in S, both of which based at x. But $N \simeq S \simeq \mathbb{R}^n \simeq *$, so these paths are homotopic to the constant path at x. Thus $\gamma \simeq c_X$, so $\pi_1(S^n) = 0$.

PROBLEM 3

Recall that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. For the torus $T = S^1 \times S^1$, this implies that

$$\pi_1(T) = \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

Let $f : T \to T$ be a continuous map. Show that one may associate to f a 2×2 integer matrix $M(f) \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ with the following properties:

(a) If $g : T \to T$ is any other continuous map, then $f \simeq g$ if and only if $M(f) = M(g)$;
(b) $M(f \circ g) = M(f) \cdot M(g)$, i.e. the matrix associated to the composition $f \circ g$ is the product of the matrices associated to f and to g.
(c) If $A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ is any 2×2 integer matrix, then there is a continuous map $f : T \to T$ such that $M(f) = A$.
(d) $f : T \to T$ is homotopy equivalent to a homeomorphism if and only if the matrix $M(f)$ is invertible.

ANSWER:

Oh boy, time to turn some topology into some algebra!! Let's look at how to define the matrix $M(f)$ first. Since $\pi_1(T) \cong \mathbb{Z}^2$, by functoriality of the fundamental group, the map f defines a group homomorphism

$$f_* : \pi_1(T) \cong \mathbb{Z}^2 \to \mathbb{Z}^2 \cong \pi_1(T).$$

We now choose a basis for $\pi_1(T) \cong \mathbb{Z}^2$ over $\mathbb{Z}$. Since $T \cong S^1 \times S^1$, we can just take the generators for the different $\pi_1(S^1)$'s as our basis. To be precise, let α be a generator for the first S^1 (think of this as a loop around the Torus going the long way around), and let β be a generator for the second S^1 (think of this as a loop going around the torus in the other way). Notice under the isomorphism $\pi_1(S^1) \cong \mathbb{Z}$, the constant map at the basepoint corresponds to 0. Then, our basis for $\pi_1(T)$ over $\mathbb{Z}$ is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

where the 1 in the first vector is generated by α, and the 1 in the second vector is generated by β.

The matrix $M(f)$ is just the induced map f_* viewed as a $\mathbb{Z}$-linear map. If $\gamma \in \pi_1(T)$ represents any loop, then we can express γ as $c\alpha + d\beta$ for some scalars $c, d \in \mathbb{Z}$. In terms of linear algebra, we can express γ as the vector $\begin{pmatrix} c \\ d \end{pmatrix}$. In this way, since f_* is a group homomorphism, we have that

$$f_*(\gamma) = f_*(c\alpha + d\beta) = f_*(c\alpha) + f_*(d\beta) = cf_*(\alpha) + df_*(\beta).$$

In terms of linear algebra, this implies that

$$f_*(\gamma) = f_* \begin{pmatrix} c \\ d \end{pmatrix} = cf_* \begin{pmatrix} 1 \\ 0 \end{pmatrix} + df_* \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, f_* is completely determined by its values on the generators α and β. By looking at the two projections $p_\alpha, p_\beta : T \to S^1$, we may determine the values of $f_*(\alpha)$ and $f_*(\beta)$ by their values in $\mathbb{Z}$ after projecting to S^1. In other words, we may represent f_* by a matrix $M(f)$ which takes the form

$$M(f) = \begin{pmatrix} f_*(\alpha) & f_*(\beta) \end{pmatrix}.$$

Thus, the matrix is the diagonal matrix formed by applying f_* to a generator, then projecting to $\mathbb{Z}$ via the map induced by the projection $T \to S^1$. The notation is a little cumbersome, but the idea is not.

For part (a), we know that $f \simeq g$ if and only if $f_* = g_*$. By the construction of the matrices above, we see then that $M(f) = M(g)$ by construction, as the values in these matrices are determined by the values of f_* and g_* on the generators of $\pi_1(T)$.

For part (b), we see:

$$M(f \circ g) = \begin{pmatrix} (f \circ g)_*(\alpha) & (f \circ g)_*(\beta) \end{pmatrix} = M(f) \cdot M(g)$$

by functoriality.

For part (c), let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be our matrix. Notice that this defines a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, i.e. we have a continuous map. Since the entries of A are integers, A must send the integers lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ to the integer lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. Notice as well that

$$\mathbb{R}^2/\mathbb{Z}^2 \cong (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \cong S^1 \times S^1 = T.$$

Thus, our matrix gives a well-defined map $T \to T$. Notice that when restricted to $S^1 \times \{0\}$, this map sends the generator α of $\pi_1(S^1)$ to the $a\alpha + b\beta \in \pi_1(T)$ (i.e. the first column of the matrix). Similarly, when restricted to $\{0\} \times S^1$, this map sends the generator β to $c\alpha + d\beta$. Thus, passing A to a self map of the torus gives a map which realizes A as a self map on fundamental groups.

For (d), just use (b) and the fact that the matrix associated to the identity map on T is the identity matrix.

PROBLEM 4

We can generalize the fundamental group to more categorical language. Recall that a groupoid is a category in which every morphism is an isomorphism.

(a) Let X be a topological space. Show that there is a groupoid $\Pi_1(X)$, called the *fundamental groupoid* of X, whose objects are the points of X and such that

$$\mathrm{Hom}_{\Pi_1(X)}(x, y) = \{[f] : f : I \to X, f(0) = x, f(1) = y\}.$$

In other words, the morphisms in $\Pi_1(X)$ are exactly the path homotopy classes of paths in X from x to y.

(b) Let $\mathcal{G}\mathrm{rpd}$ denote the category whose objects are groupoids and morphisms are functors. Show that the fundamental groupoid defined above describes a functor $\Pi_1 : \mathcal{T}\mathrm{op} \to \mathcal{G}\mathrm{rpd}$.

ANSWER:

For (a), to show that $\Pi_1(X)$ is a groupoid, we must first show that the objects and morphisms described above actually form a category. In fact, we have already done this in our first time working homotopies. Composition is given by path concatenation, and we showed that path concatenation is an associative process. We also showed that composition with the identity on either side does nothing, i.e. if $[f] \in \mathrm{Hom}_{\Pi_1(X)}(x, y)$, then

$$[f] \cdot [c_x] = [c_y] \cdot [f] = [f].$$

Thus $\Pi_1(X)$ is a cateogory. To show that it is a groupoid, we must show that every morphism is an isomorphism. This is also straightforward: if $f : I \to X$ is any path from x to y, then there is the reverse path $f^{\mathrm{rev}} : I \to X$, and have shown that

$$[f] \cdot [f^{\mathrm{rev}}] = [c_y], \quad [f^{\mathrm{rev}}] \cdot [f] = [c_x].$$

Thus $\Pi_1(X)$ is a groupoid.

For (b), if we take the identity map $\mathrm{id}_X : X \to X$ for a space $X \in \mathrm{ob}(\mathcal{T}\mathrm{op})$, then there is an induced map of fundamental groupoids $\Pi_1(\mathrm{id}_X) : \Pi_1(X) \to \Pi_1(X)$. Notice that a morphism of groupoids is just a functor, and this induced functor just postcomposes any path in X (i.e. any morphism in $\Pi_1(X)$) with the identity map. In particular, every morphism in $\Pi_1(X)$ is an isomorphism, and so is id_X, hence the composition is as well. Moreover, since path composition is associative, Π_1 respects function composition. Thus Π_1 defines a functor.