MATH 480: HOMOTOPY THEORY HOMEWORK 2

ABSTRACT. Homework 2 due in class on **Wednesday, April 15**.

1. PROBLEMS

The first problem on this homework set is a fun application of the computation of $\pi_1(S^1) \cong \mathbb{Z}$.

(1) (**Fundamental Theorem of Algebra**) Let $f(x)$ be the complex polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where a_i are not all zero. Then there exists some complex number $z \in \mathbb{C}$ such that $f(z) = 0$. (**Hint:** for sake of contradiction, suppose that $f(x) \neq 0$ for any $x \in \mathbb{C}$. Viewing $S^1 \subseteq \mathbb{C}$, this allows one to define a continuous function $\hat{f} : S^1 \to S^1$ by

$$\hat{f}(x) = \frac{f(x)}{||f(x)||}.$$

Use $\hat{f}$ to contradict $\pi_1(S^1) \cong \mathbb{Z}$.)

(2) Let X be some space which is the unions of some path connected open subsets $\{U_\alpha\}$, each of which containing the basepoint $x_0 \in X$ (i.e. $x_0 \in U_\alpha$ for all α), and such that each pairwise intersection $U_{\alpha_1} \cap U_{\alpha_2}$ is path-connected. Prove that if $\gamma : I \to X$ is any loop in X based at x_0, then there are loops $\gamma_\alpha : I \to U_\alpha$ based at x_0 for every U_α such that

$$\gamma \simeq \gamma_{\alpha_1} \cdot \gamma_{\alpha_2} \cdot \gamma_{\alpha_2} \cdot \cdots$$

In other words, any element $\gamma \in \pi_1(X, x_0)$ can be rewritten as a product of loops $\gamma_\alpha \in \pi_1(U_\alpha, x_0)$ for each U_α. As a consequence, show that $\pi_1(S^n) = 0$ for $n \geq 2$.

(3) Recall that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. For the torus $T = S^1 \times S^1$, this implies that

$$\pi_1(T) = \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

Let $f : T \to T$ be a continuous map. Show that one may associate to f a 2×2 integer matrix $M(f) \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ with the following properties:

(a) If $g : T \to T$ is any other continuous map, then $f \simeq g$ if and only if $M(f) = M(g)$;

(b) $M(f \circ g) = M(f) \cdot M(g)$, i.e. the matrix associated to the composition $f \circ g$ is the product of the matrices associated to f and to g.

(c) If $A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ is any 2×2 integer matrix, then there is a continuous map $f : T \to T$ such that $M(f) = A$.

(d) $f : T \to T$ is homotopy equivalent to a homeomorphism if and only if the matrix $M(f)$ is invertible.

(4) We can generalize the fundamental group to more categorical language. Recall that a groupoid is a category in which every morphism is an isomorphism.

(a) Let X be a topological space. Show that there is a groupoid $\Pi_1(X)$, called the *fundamental groupoid* of X, whose objects are the points of X and such that

$$\mathrm{Hom}_{\Pi_1(X)}(x, y) = \{[f] : f : I \to X,\ f(0) = x,\ f(1) = y\}.$$

In other words, the morphisms in $\Pi_1(X)$ are exactly the path homotopy classes of paths in X from x to y.

(b) Let $\mathcal{G}\mathrm{rpd}$ denote the category whose objects are groupoids and morphisms are functors. Show that the fundamental groupoid defined above describes a functor $\Pi_1 : \mathcal{T}\mathrm{op} \to \mathcal{G}\mathrm{rpd}$.